

Convex Powerdomains I

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A completion via Frink ideals is used to define a convex powerdomain of an arbitrary continuous lattice as a continuous lattice. The powerdomain operator is a functor in the category of continuous lattices and continuous inf-preserving maps and preserves projective limits and surjectivity of morphisms; hence one can solve domain equations in which it occurs. Analogous results hold for algebraic lattices and bounded complete algebraic cpo's. © 1987 Academic Press, Inc.

The motivation for the study of powerdomains comes from denotational semantics for programming languages. Mathematical foundations for the denotational approach to semantics were developed by Dana Scott in a series of fundamental papers including (Scott, 1970, 1972, 1976, 1981, 1982). The basic idea is to represent data types by certain partially ordered sets (posets) called *domains* and computations by certain functions from domains into domains called *continuous maps*. In order to be capable of describing the usual programming constructs, the category of domains and continuous maps has to be closed under sum $+$, product \times , their strict variants \oplus , \otimes , the function space construct \rightarrow , etc., and it must allow for solutions of *domain equations* $D \cong T(D)$, where T is an expression built up from $+$, \times , \oplus , \otimes , \rightarrow , etc.

In Scott's original approach domains were *continuous lattices* (see Scott, 1972); later it turned out that the classical *algebraic lattices* could be used as well (Scott, 1976). The presence of the top element 1 in lattices is rather unnatural for some applications in denotational semantics; for example, there are very simple programming constructs with natural equational characterizations that cannot be consistently extended to the top element (this was kindly brought to my attention by G. Plotkin). Hence many people prefer the category of *bounded complete* (consistently complete) *algebraic cpo's* (essentially algebraic lattices with the top element removed), which avoids these problems while retaining all the pleasing properties of the category of algebraic lattices. These are the objects that Scott calls domains in his latter papers (Scott, 1981, 1982). Experience shows that they are adequate for modelling all sequential deterministic programming language constructs.

The notion of a *powerdomain* was invented in order to extend Scott's framework to the simplest nondeterministic features of programming languages. The original Egli–Milner construction works only for flat domains; Plotkin (1976) showed how to define *powerdomains of arbitrary ω -algebraic cpo's* (Smyth (1978, 1983) streamlined Plotkin's original approach). Unfortunately the category of algebraic cpo's is not cartesian closed (it is not closed under $\rightarrow!$), and hence it is not suitable for the purposes of denotational semantics; on the other hand, the category of bounded complete algebraic cpo's (or algebraic lattices) is not closed under the Plotkin powerdomain construction.

Plotkin resolved the above-mentioned difficulties by way of defining a category of *SFP-objects*, larger than the category of all bounded complete ω -algebraic cpo's, but smaller than the category of all ω -algebraic cpo's. This category of SFP-objects is cartesian closed, and it is also closed under the Plotkin powerdomain. Its great drawback is that SFP-objects are considerably more complicated and harder to work with than algebraic lattices or bounded complete algebraic cpo's.

Plotkin suggested another possibility—completing SFP-objects into algebraic lattices—but dismissed it as impractical. On page 463 in Plotkin (1976) he writes, “So converting $P(D)$ into a lattice would require one to add many points—not just a top element. It is not clear to the author how to keep these separate from the bona-fide elements.” And again on p. 482, “we can embed any SFP-object in $\mathcal{P}(\omega)$... and this gives rise to a lattice with intermediate points. But these intermediate points seem to clutter up the domain... what is wanted is a simple development of $P(\cdot)$ in the context of $\mathcal{P}(\omega)$ or a similar simple structure. In Scott's words, we want an analytic, not a synthetic, development.”

We shall try to do just that in this paper. It appears that our construction behaves quite nicely both in its mathematical properties and from the point of view of applications to the description of nondeterminism. The Plotkin powerdomain is embedded as a cofinal subset into this construction, elements of the Plotkin powerdomain can be easily distinguished from the “new” elements, and the “new” elements can be given a meaningful intuitive interpretation. In addition, our construction generalizes to the category of continuous lattices (or bounded complete continuous cpo's), and we do not need to assume existence of a countable basis.

We shall state and prove all our results for the categories of continuous and algebraic lattices (and continuous maps). Analogous results for bounded complete continuous cpo's (= complete-continuous semilattices of Gierz *et al.*, 1980) and bounded complete algebraic cpo's require only straightforward modifications and will not be commented upon any further. Our first report on this work, Hrbacek (1985), is in the setting of bounded complete algebraic cpo's.

We mention in passing that, in addition to the Plotkin powerdomain, also known as the *convex powerdomain*, other related species of powerdomains have been studied. Smyth (1978) introduced the *Smyth* (or *upper*) *powerdomain*, and there is an analogous *Hoare* (or *lower*) *powerdomain* (see Plotkin, 1981). These do not present any theoretical difficulties but provide only a very much coarser description of nondeterministic computations. In the rest of the paper the word *powerdomain* will refer to a Plotkin-style convex powerdomain based on Egli–Milner ordering.

The present paper grew out of my interest in the mathematical foundations of denotational semantics sparked by the lectures of J. Stoy at the International Summer School in Marktoberdorf in 1981. A number of people helped by lending their ear to the reports on various stages of the work. My particular thanks go to Professor Gordon Plotkin for a number of penetrating observations on the first draft of Hrbacek (1985) that have greatly influenced my further thinking about the subject; his lecture notes Plotkin (1981) have also been very useful.

1. PRELIMINARIES

This section introduces terminology and notation, and surveys some basic facts needed in the rest of the paper. We shall follow Gierz *et al.* (1980) as closely as possible; this is the source one should consult for definitions of unexplained concepts, proofs, and much other related information.

Let (D, \leq) be a partially ordered set (*poset*); we always assume $D \neq \emptyset$. We say that $a \in D$ is a *lower* [*upper*] *bound* of $X \subseteq D$ and write $a \leq X$ [$X \leq a$] if $a \leq x$ [$x \leq a$] for all $x \in X$; we denote the set of all lower [*upper*] bounds of X by $X\downarrow$ [$X\uparrow$]. The *least upper bound* (*supremum*) of X will be denoted $\bigvee X$ or $\sup X$, if it exists. Similarly for the *greatest lower bound* (*infimum*) $\bigwedge X$ or $\inf X$. We denote the least and the greatest elements of D by 0 and 1, respectively (if they exist).

A set $X \subseteq D$ is *directed* if every finite subset of X has an upper bound in X ; so every directed set is nonempty. We let $\downarrow X = \{y \in D \mid y \leq x \text{ for some } x \in X\}$ and $\uparrow X = \{y \in D \mid x \leq y \text{ for some } x \in X\}$. X is a *lower set* if $X = \downarrow X$ and an *upper set* if $X = \uparrow X$. If $x \in D$, $\langle x \rangle = \downarrow\{x\} = \{x\}\downarrow = \{y \in D \mid y \leq x\}$.

Directed ideals of (D, \leq) are directed lower sets; they are called simply *ideals* in Gierz *et al.* (1980), but a more general notion of an ideal, introduced in Section 2, will be essential to our approach. $|D|$ is the set of all directed ideals of (D, \leq) . For any $x \in D$, $\langle x \rangle \in |D|$ is the *principal ideal* generated by x .

Let (D, \leq_D) and (E, \leq_E) be posets; a function $f: D \rightarrow E$ is *monotone* (or *order-preserving*) if $x \leq_D y$ implies $f(x) \leq_E f(y)$. An *embedding* is a 1–1 function $f: D \rightarrow E$ such that $x \leq_D y \Leftrightarrow f(x) \leq_E f(y)$. If f is also onto E , it is

called an *isomorphism*. We say that a monotone $f: D \rightarrow E$ *preserves* [finite, directed] *sup*s if whenever X is a [finite, directed] subset of D and $\sup^D X$ exists, then $\sup^E f[X]$ exists and $\sup^E f[X] = f(\sup^D X)$. Here, of course, $f[X] = \{f(x) \mid x \in X\}$. Similarly, f *preserves* *inf*s if $\inf^E f[X] = f(\inf^D X)$ whenever $\inf^D X$ exists.

A poset (D, \leq) is called *up-complete* or a *complete partial order* (cpo) if every directed $X \subseteq D$ has a least upper bound $\sup X \in D$. If (D, \leq_D) and (E, \leq_E) are cpo's and $f: D \rightarrow E$ preserves directed sups, f is called *continuous*. The set of all continuous functions from D into E is denoted $[D \rightarrow E]$; it is well known that it is also a cpo if one defines $f \leq g \Leftrightarrow (\forall x \in D) (f(x) \leq_E g(x))$. $1_D \in [D \rightarrow D]$ is the *identity function* from D to D .

A crucial role in singling out cpo's of particular interest to the theory of domains is played by the next definition. Let (D, \leq) be a cpo. We say that a is *way-below* b (notation: $a \ll b$) if for every directed $X \subseteq D$, $b \leq \sup X$ implies $a \leq x$ for some $x \in X$. An element $a \in D$ is *compact* if $a \ll a$ (i.e., for any directed $X \subseteq D$, $a \leq \sup X$ implies $a \leq x$ for some $x \in X$). $K(D)$ will denote the set of all compact elements of D .

PROPOSITION 1. *In any cpo (D, \leq) the relation \ll has the following properties:*

- (P1) $x \ll y$ implies $x \leq y$.
- (P2) $u \leq x \ll y \leq z$ implies $u \ll z$.
- (P3') $0 \ll x$ if D has the least element.
- (P4') If $x \ll z$ and $y \ll z$ then $x \vee y \ll z$ if it exists.

Proof. See Gierz *et al.* (1980, I, Proposition 1.2, and the remarks preceding Definition 1.26). ■

We let $\downarrow X = \{y \in D \mid y \ll x \text{ for some } x \in X\}$. A cpo (D, \leq) is called *continuous* if, for all $x \in D$, the set $\downarrow\{x\} = \{y \in D \mid y \ll x\}$ is directed and $x = \sup \downarrow\{x\}$. It is called *algebraic* if, for all $x \in D$, the set $\downarrow\{x\} \cap K(D)$ of all compact elements below x is directed and $x = \sup \downarrow\{x\} \cap K(D)$. It is *ω -algebraic* if, in addition, $K(D)$ is at most countable.

PROPOSITION 2. *In any continuous cpo (D, \leq) the relation \ll has the following additional properties:*

- (P3) For every $x \in D$ there is $y \ll x$.
- (P4) If $x \ll z$ and $y \ll z$ then there is $u \ll z$ such that $x \leq u$ and $y \leq u$.
- (P5) If $x \not\ll y$ then there is $z \in D$ such that $z \not\ll y$ and $z \ll x$.
- (INT) If $x \ll y$ then $x \ll z \ll y$ for some $z \in D$ (the *interpolation property*).

Proof. (P3), (P4), and (P5) are immediate consequences of continuity. For example, to prove (P5) we note that $z \leq y$ for all $z \ll x$ implies $x = \sup\{z \in D \mid z \ll x\} \leq y$. For the proof of (INT) see Gierz *et al.* (1980, Exercise 1.27 and Lemmas 1.16 and 1.17). ■

As an immediate consequence of (INT), the following holds in every continuous cpo (D, \leq) : if $x \ll y$, $X \subseteq D$ is directed, and $y \leq \sup X$, then $x \ll z$ for some $z \in X$.

A poset (D, \leq) is a *sup-semilattice with 0* if every finite subset of D has a least upper bound. Sup-semilattices with 0 which are at the same time continuous cpo's are the *continuous lattices* of Scott (1972), see also Gierz *et al.* (1980); they and the well-known subclass of *algebraic lattices* (i.e., algebraic cpo's, where every finite subset has a least upper bound) will be the main objects of our study. We next survey some well-known methods for obtaining algebraic and continuous cpo's and lattices via ideal completions.

PROPOSITION 3. *Let (D, \leq) be a poset. The set $|D|$ of all directed ideals, ordered by inclusion, is an algebraic cpo. The function $\langle \cdot \rangle$ defined by $\langle x \rangle = \downarrow\{x\}$ is an embedding of (D, \leq) into $(|D|, \subseteq)$ and preserves finite sups. The principal ideals of D are precisely the compact elements of $(|D|, \subseteq)$. If (D, \leq) is a sup-semilattice with 0, then $(|D|, \subseteq)$ is an algebraic lattice. $(|D|, \subseteq)$ is called the ideal completion of (D, \leq) .*

Proof. For algebraic lattices this is a classical result that goes back to Birkhoff; see Grätzer (1979, Chap.0), or Gierz *et al.* (1980, Proposition 4.12). The generalization to algebraic posets is straightforward. ■

Proposition 3 has a generalization to continuous posets (Smyth, 1978; Gierz *et al.*, 1980, I, 1, 2, and III, 4; Gierz and Keimel, 1981). We call a binary relation $<$ on a poset (D, \leq) *auxiliary* if it has the properties (P1) and (P2) from Proposition 1, and (P3)–(P5) and (INT) from Proposition 2, with $<$ in place of \ll . (In Gierz *et al.*, 1980 (P5) is not required, and much of what follows holds true without it.) A directed ideal I is called *round* if for every $x \in I$ there is some $y \in I$ such that $x < y$. For every directed ideal I let $c(I) = \{x \in D \mid x < y \text{ for some } y \in I\}$. We note that I is round if and only if $c(I) = I$.

PROPOSITION 4. *Let I and J be directed ideals of a poset (D, \leq) with an auxiliary relation $<$.*

- (1) $c(I)$ is a directed ideal.
- (2) $c(I) \subseteq I$.
- (3) $I \subseteq J$ implies $c(I) \subseteq c(J)$.

- (4) c preserves directed sups (as a map of $(|D|, \subseteq)$ into itself).
- (5) $c(I) = c(c(I))$; hence $c(I)$ is round.

Proof. Straightforward; see Gierz and Keimel (1981) in case (D, \leq) is a sup-semilattice with 0. ■

In other words, c is a kernel operator on the algebraic poset $(|D|, \subseteq)$ and preserves directed sups. We now get

PROPOSITION 5. *Let (D, \leq) be a poset with an auxiliary relation \prec . The set $\|D\|$ of all round directed ideals, ordered by inclusion, is a continuous cpo. The function $\ll \cdot \gg$ defined by $\ll x \gg = c(\langle x \rangle)$ is an embedding of (D, \leq) into $(\|D\|, \subseteq)$. For $x, y \in D$, $x \prec y \Leftrightarrow \ll x \gg \in \ll y \gg$ where \in is the way-below relation in $(\|D\|, \subseteq)$. In general, for $I, J \in \|D\|$, $I \in J \Leftrightarrow I \subseteq \ll x \gg$ for some $x \in J$. If (D, \leq) is a sup-semilattice with 0 then $(\|D\|, \subseteq)$ is a continuous lattice. We shall call $(\|D\|, \subseteq)$ the continuous completion of (D, \leq, \prec) . If \prec is \leq , $(\|D\|, \subseteq)$ is just the ideal completion $(|D|, \subseteq)$.*

Proof. These ideas are essentially due to Smyth (1978). The proof for sup-semilattices with 0 can be found in Gierz *et al.* (1980) and in Gierz and Kiemel (1981). ■

We remark that $\ll \cdot \gg$ need not preserve finite sups; it will if (D, \leq, \prec) satisfies an additional condition: $z \prec x \vee y \Rightarrow (\exists u \prec x) (\exists v \prec y) z \leq u \vee v$. This condition holds whenever D is a basis for a continuous lattice and \prec is its way-below relation.

In the sequel, **CPO** will be the category of all complete partial orders and all continuous functions between them. The following full subcategories of **CPO** will be of particular interest: **CP** (continuous cpo's), **AP** (algebraic cpo's), **CONT** (continuous lattices), and **ALG** (algebraic lattices). The category **CL** of all continuous lattices and all continuous inf-preserving functions and its full subcategory **AL** of all algebraic lattices will also be important, especially in Section 3.

2. POWERDOMAINS

Let (D, \leq) be a poset; $P(D)$ will denote the set of all *nonempty finite* subsets of D . For $X, Y \in P(D)$ we define

$$X \leq_{\text{EM}} Y \Leftrightarrow (\forall x \in X) (\exists y \in Y) x \leq y \text{ and } (\forall y \in Y) (\exists x \in X) x \leq y;$$

\leq_{EM} is called the *Egli-Milner preorder* on $P(D)$; it is clearly reflexive and transitive but usually not antisymmetric. One can obtain a poset by

forming equivalence classes of $P(D)$ modulo the equivalence relation $X \equiv_{EM} Y \Leftrightarrow X \leq_{EM} Y \ \& \ Y \leq_{EM} X$. If \bar{X} , \bar{Y} are the equivalence classes of X and Y modulo \equiv_{EM} , we let $\bar{X} \leq_{EM} \bar{Y} \Leftrightarrow X \leq_{EM} Y$.

In order not to clutter up our notation and terminology, we shall work with the representatives X , Y , ..., rather than their equivalence classes \bar{X} , \bar{Y} , ..., and talk about the poset $(P(D), \leq_{EM})$, its ideals, etc., when we really mean $(\{\bar{X} \mid X \in P(D)\}, \leq_{EM})$, etc.; we shall revert to equivalence classes if necessary to avoid misunderstandings.

We can now define powerdomains in the category **AP** (Plotkin, 1976; Smyth, 1978). Let (D, \leq) be an algebraic cpo and let $K(D)$ be the set of its compact elements. The *powerdomain* of (D, \leq) is the ideal completion of $(P(K(D)), \leq_{EM})$; we denote it $P^{AP}(D, \leq)$. By Proposition 3 in Section 1, $P^{AP}(D)$ is again an algebraic cpo.

The category **AP** is not suitable for use as a category of domains in denotational semantics, because it is not cartesian closed; that is, $[D \rightarrow E]$ need not be an algebraic cpo for D , E algebraic (Markowsky and Rosen, 1976). On the other hand, the category **ALG**, while cartesian closed, is not closed under P^{AP} .

We propose to define the powerdomain of an algebraic lattice (D, \leq) in such a way that it is again an algebraic lattice. This can be achieved by using a different completion of $(P(K(D)), \leq_{EM})$, one which yields an algebraic lattice rather than merely an algebraic cpo. There are, of course, many such completions; in this paper we employ the one based on a notion of ideal due to Frink (1954). This choice has been motivated by the well-known fact (see, e.g., Ern , 1981) that the Frink completion is *minimal*; this is stated precisely in the next proposition, which we give without proof.

PROPOSITION. *Let (D, \leq) be a poset and $(|D|_F, \subseteq)$ its Frink completion (as defined below). Let (E, \leq_E) be any algebraic lattice such that $D \subseteq K(E)$ and for every $e \in E$, $e = \sup_E \{d \in D \mid d \leq_E e\}$. Then there is a unique continuous inf-preserving f and a unique sup-preserving g such that the following diagram commutes:*

$$\begin{array}{ccc} D & \xrightarrow{\subseteq} & E \\ \downarrow \langle \cdot \rangle & \nearrow g & \nearrow f \\ |D|_F & & \end{array}$$

Moreover, (f, g) is an adjunction, f is injective, and g is surjective.

Hence Frink completion will introduce as few “extraneous” (i.e., not in $P^{AP}(D)$) elements as possible. Moreover, these extraneous elements can be given an intuitive “computational” meaning, as shown by Theorem 4 in

Section 4 of Hrbacek (submitted) and the subsequent discussion; this last observation was in fact the starting point of the present work.

We now proceed to define the Frink completion. Let (D, \leq) be any poset ($D \neq \emptyset$). A set $I \subseteq D$ is an *ideal* (more precisely, *Frink ideal*) if for every finite $F \subseteq I$ and every $x \in D$, $x \leq F \uparrow$ implies $x \in I$. $I = \emptyset$ is allowed, but if D has the least element 0 then all ideals of D are nonempty. If every finite subset of I is bounded in D , I is called *proper*; so I is not proper only if $I = D$. All directed ideals are Frink ideals, and the converse is true in sup-semilattices with 0, but not in general. An intersection of any set of ideals and a union of any \subseteq -directed set of ideals is an ideal. Thus the set $|D|_F$ of all ideals of D is an *algebraic closure system*. For any $X \subseteq D$ we let $[X] = \bigcap \{I \in |D|_F \mid X \subseteq I\} = \{z \in D \mid z \leq F \uparrow \text{ for some finite } F \subseteq X\}$. We call $[X]$ the *ideal generated* by X and note that $[\{x\}] = \langle x \rangle$. Lemmas 3, 4, and 5 and Theorem 4 in Grätzer (1979, Chap. 0, Section 6), now yield the well-known result crucial to our approach.

THEOREM 1. *Let (D, \leq) be a poset. The set $|D|_F$ of all ideals of D , ordered by inclusion, is an algebraic lattice. The function $\langle \cdot \rangle$ is an embedding of (D, \leq) into $(|D|_F, \subseteq)$ and preserves finite sups. An ideal I is a compact element of $(|D|_F, \subseteq)$ if and only if it is finitely generated, i.e., $I = [X] = \sup\{\langle x \rangle \mid x \in X\}$ for some finite $X \subseteq D$. We call $(|D|_F, \subseteq)$ the Frink completion of (D, \leq) . ■*

Let (D, \leq) be an algebraic lattice; we define its *powerdomain* $\mathbf{P}^{\mathbf{AL}}(D, \leq)$ as the Frink completion of $(P(K(D)), \leq_{\mathbf{EM}})$. By Theorem 1, $\mathbf{P}^{\mathbf{AL}}(D)$ is again an algebraic lattice. We note that $\mathbf{P}^{\mathbf{AP}}(D) \subseteq \mathbf{P}^{\mathbf{AL}}(D)$. (We shall see in Section 3 that $\mathbf{P}^{\mathbf{AL}}$ is a functor in the category \mathbf{AL} , but not \mathbf{ALG} —hence the notation.)

This construction can be generalized to continuous lattices. To prepare the ground, we first develop a Plotkin-style powerdomain construction for continuous cpo's. The idea is to use Proposition 5 in place of Proposition 3. For another approach to the construction of powerdomains for continuous cpo's see Kamimura and Tang (1984).

Let (D, \leq) be a poset with an auxiliary relation $<_{\mathbf{EM}}$. There is an obvious candidate for an auxiliary relation on $(P(D), \leq_{\mathbf{EM}})$: for $X, Y \in P(D)$ let $X <_{\mathbf{EM}} Y \Leftrightarrow (\forall x \in X) (\exists y \in Y) x < y$ and $(\forall y \in Y) (\exists x \in X) x < y$.

THEOREM 2. *$<_{\mathbf{EM}}$ is an auxiliary relation on $(P(D), \leq_{\mathbf{EM}})$.*

Proof. (P1), (P2), and (P3) are obvious.

(P4) Let $X <_{\mathbf{EM}} Z$ and $Y <_{\mathbf{EM}} Z$. For each triple (x, y, z) such that $x \in X$, $y \in Y$, $z \in Z$ and $x < z$, $y < z$, there is some $u \in D$ such that $x \leq u$,

$y \leq u$, and $u < z$ (by (P4) for $<$); we put one such u into U for each such triple. Then $U \in P(D)$ and it is easy to check that $X \leq_{EM} U$, $Y \leq_{EM} U$, and $U <_{EM} Z$.

(INT) Let $X <_{EM} Y$. For each pair (x, y) such that $x \in X$, $y \in Y$, and $x < y$ there is some $z \in D$ for which $x < z < y$, by (INT) for $<$. We put one such z into Z for each such pair; it follows that $X <_{EM} Z <_{EM} Y$, as required.

(P5) Assume $X \not\leq_{EM} Y$. There are two cases:

(i) $(\exists y \in Y) (\forall x \in X) x \not\leq y$. Fix such y . By (P5) for $<$ we can find, for each $x \in X$, some $z_x < x$, $z_x \not\leq y$. Let $Z = \{z_x | x \in X\}$. Then $Z <_{EM} X$ and $Z \not\leq_{EM} Y$.

(ii) $(\exists x \in X) (\forall y \in Y) x \not\leq y$. Fix such $x = \bar{x}$. Again, for each $y \in Y$ there is some z_y such that $z_y < \bar{x}$, $z_y \not\leq y$. By (P4) for $<$, there is some $u < \bar{x}$ such that $z_y \leq u$ for all $y \in Y$. It follows that $u < \bar{x}$ and $u \not\leq y$ for any $y \in Y$. For every $x \in X$, $x \neq \bar{x}$, pick some $z_x < x$ (by (P3)) and let $Z = \{u\} \cup \{z_x | x \in X, x \neq \bar{x}\}$. Clearly $Z <_{EM} X$ and $Z \not\leq_{EM} Y$ (because $(\forall y \in Y) u \not\leq y$). ■

We can now define powerdomains in the category **CP**: if (D, \leq) is a continuous cpo, $\mathbf{P}^{\mathbf{CP}}(D, \leq)$ is the continuous completion of $(P(D), \leq_{EM}, <_{EM})$, where the way-below relation \ll on D is used as $<$.

THEOREM 3. *If (D, \leq) is a continuous cpo then $\mathbf{P}^{\mathbf{CP}}(D)$ is a continuous cpo. For $X, Y \in P(D)$, $X <_{EM} Y \Leftrightarrow \langle\langle X \rangle\rangle \in \langle\langle Y \rangle\rangle$. In general, for $\mathcal{J}, \mathcal{J}' \in \|P(D)\|$, $\mathcal{J} \in \mathcal{J}' \Leftrightarrow \mathcal{J} \subseteq \langle\langle X \rangle\rangle$ for some $X \in \mathcal{J}$.*

Proof. Immediate from Propositions 1 and 2, Theorem 2, and Proposition 5. ■

THEOREM 4. *Let (D, \leq) be an algebraic cpo. Then there is a unique isomorphism k between $\mathbf{P}^{\mathbf{AP}}(D)$ and $\mathbf{P}^{\mathbf{CP}}(D)$ such that $k(\langle x \rangle) = \langle\langle X \rangle\rangle = \langle X \rangle$ holds for all $X \in P(K(D))$.*

Proof. We omit the proof, as it is quite similar to the proof of Theorem 13, given in detail later. ■

Finally, we want to define a powerdomain of a continuous lattice D that would bear the same relationship to $\mathbf{P}^{\mathbf{CP}}(D)$ as $\mathbf{P}^{\mathbf{AL}}(D)$ bears to $\mathbf{P}^{\mathbf{AP}}(D)$ for algebraic lattices. To do so, we need to generalize the notion of roundness to Frink ideals.

Let (D, \leq) be a poset with an auxiliary relation $<$. We say that a (Frink) ideal I of D is *round* if for every $x \in I$ there exists a finite $Y \subseteq I$ such that $x < Y \uparrow$ (i.e., $x < z$ for every upper bound z of Y). For directed ideals

this is equivalent to the original definition. For every ideal I , let $c(I) = \{x \in D \mid x < Y \uparrow \text{ for some finite } Y \subseteq I\}$. Note that if I is directed, $c(I) = \{x \in D \mid x < y \text{ for some } y \in I\}$, as before. Also, I is round if and only if $I = c(I)$. We now need an analog of Proposition 4 in Section 1; this seems to require an additional assumption on $<$. We say that $<$ is a *strong auxiliary relation* if it has the *strong interpolation property*

(SINT) If $x < Y \uparrow$ for some finite Y , then there is a finite Z such that $x < Z \uparrow$ and $z < Y \uparrow$ for all $z \in Z$.

We say that $<$ has the *very strong interpolation property* if

(VSINT) If $x < Y \uparrow$ for some finite $Y = \{y_0, \dots, y_n\}$, then there exists $Z = \{z_0, \dots, z_n\}$ such that $x < Z \uparrow$ and $z_i < y_i$ for all i , $0 \leq i \leq n$.

It is easy to see that the way-below relation \ll in any continuous lattice has (VSINT); a less trivial example is provided by Theorem 7. But first we prove

THEOREM 5. *Let I, J be ideals of a poset (D, \leq) with a strong auxiliary relation $<$. Then*

- (1) $c(I)$ is an ideal.
- (2) $c(I) \subseteq I$.
- (3) $I \subseteq J$ implies $c(I) \subseteq c(J)$.
- (4) c preserves directed sups.
- (5) $c(I) = c(c(I))$; hence $c(I)$ is round.

Proof. (1) Let $x \leq F \uparrow$ for some finite $F \subseteq c(I)$. By definition of $c(I)$, for each $z \in F$ there is a finite $Y_z \subseteq I$ such that $z < Y_z \uparrow$. Let $Y = \bigcup \{Y_z \mid z \in F\}$; then $Y \subseteq I$ is finite. If $y \in Y \uparrow$ then $y \in Y_z \uparrow$ and so $z < y$, for all $z \in F$. By (P4) there is a $u < y$ such that $u \in F \uparrow$; and hence $x \leq u < y$. Thus $x < Y \uparrow$ for finite $Y \subseteq I$, and so $x \in c(I)$.

(2) $x < Y \uparrow$ implies $x \leq Y \uparrow$ by (P1), so $c(I) \subseteq I$ follows immediately from the definition.

(3) This is trivial.

(4) Let \mathcal{J} be an \subseteq -directed set of ideals and let $J = \sup \mathcal{J} = \bigcup \mathcal{J}$. Clearly $I \in \mathcal{J}$ implies $c(I) \subseteq c(J)$ by (3), so $\bigcup \{c(I) \mid I \in \mathcal{J}\} \subseteq c(J)$. Conversely, let $x \in c(J)$; then $x < Y \uparrow$ for some finite $Y \subseteq J = \bigcup \mathcal{J}$. Since \mathcal{J} is \subseteq -directed, $Y \subseteq I$ for some $I \in \mathcal{J}$ and so $x \in c(I)$. This shows $c(J) \subseteq \bigcup \{c(I) \mid I \in \mathcal{J}\}$ and hence $c(J) = \sup \{c(I) \mid I \in \mathcal{J}\}$.

(5) $c(c(I)) \subseteq c(I)$ follows immediately from (2). For the converse, let $x \in c(I)$. Then $x < Y \uparrow$ for some finite $Y \subseteq I$. Thus by (SINT) there is a finite Z such that $x < Z \uparrow$ and $z < Y \uparrow$ for all $z \in Z$, i.e., $Z \subseteq c(I)$. Thus $x \in c(c(I))$. ■

For any $X \subseteq D$ we let $\llbracket X \rrbracket = c(\llbracket X \rrbracket) = \{z \in D \mid z \prec F\uparrow \text{ for some finite } F \subseteq X\}$ be the *round ideal generated by X*. We have to prove the last equality. If $z \in c(\llbracket X \rrbracket)$ then $z \prec T\uparrow$ for some finite $T \subseteq \llbracket X \rrbracket$. By definition of $\llbracket X \rrbracket$, for each $t \in T$ there is a finite $F_t \subseteq X$ such that $t \leq F_t\uparrow$. Let $F = \bigcup \{F_t \mid t \in T\}$; then $F \subseteq X$ is finite and for every $t \in T$, $t \leq F\uparrow$. Thus $F\uparrow \subseteq T\uparrow$ and $z \prec F\uparrow$. Conversely, if $z \prec F\uparrow$ for $F \subseteq X \subseteq \llbracket X \rrbracket$ finite, then $z \in c(\llbracket X \rrbracket)$. Clearly $\llbracket X \rrbracket = \{z \in D \mid z \prec X\uparrow\}$ if X is finite, and $\llbracket \{x\} \rrbracket = \llbracket x \rrbracket = \{z \in D \mid z \prec x\}$. We now have an analog of Theorem 1.

THEOREM 6. *Let (D, \leq) be a poset with a strong auxiliary relation \prec . The set $\|D\|_F$ of all round ideals of D , ordered by inclusion, is a continuous lattice. The function $\llbracket \cdot \rrbracket$ is an embedding of (D, \leq) into $(\|D\|_F, \subseteq)$. For $x, y \in D$, $x \prec y \Leftrightarrow \llbracket x \rrbracket \in \llbracket y \rrbracket$. In general, for $I, J \in \|D\|_F$, $I \in J \Leftrightarrow I \subseteq \llbracket X \rrbracket$ for some finite $X \subseteq J$. We shall call $(\|D\|_F, \subseteq)$ the continuous Frink completion of (D, \leq) . If \prec has the very strong interpolation property, then $\llbracket X \rrbracket = \sup\{\llbracket x \rrbracket \mid x \in X \text{ for all } X \subseteq D; \text{ in particular, } \llbracket \cdot \rrbracket \text{ preserves finite sups.}$*

Proof. By Theorem 1, $(\|D\|_F, \subseteq)$ is an algebraic lattice. Theorem 5 states, in effect, that c is a kernel operator on $\|D\|_F$ and preserves directed sups. By Lemma 2.10 in Gierz *et al.* (1980, I, 2), the image of $\|D\|_F$ under c (i.e., $\llbracket \cdot \rrbracket$) is a continuous lattice. The statements concerning $\llbracket \cdot \rrbracket$ are quite straightforward.

Let $I \in J$; $J = \bigcup \{\llbracket X \rrbracket \mid X \subseteq J, X \text{ finite}\}$: for every $x \in J$ there exists a finite $X \subseteq J$ such that $x \prec X\uparrow$ by roundness. By definition of $\llbracket \cdot \rrbracket$, $I \subseteq \llbracket X \rrbracket$ for some such X .

Conversely, let $I \subseteq \llbracket X \rrbracket$ for some finite $X \subseteq J$. If $J = \bigcup \mathcal{K}$, where \mathcal{K} is an \subseteq -directed system of round ideals, then there is some $K \in \mathcal{K}$ such that $X \subseteq K$ and so $I \subseteq \llbracket X \rrbracket \subseteq K$. This shows $I \in J$.

Let $X \subseteq D$; clearly $\llbracket x \rrbracket \subseteq \llbracket X \rrbracket$ for all $x \in X$, so $I = \sup\{\llbracket x \rrbracket \mid x \in X\} \subseteq \llbracket X \rrbracket$. On the other hand, $z \in \llbracket X \rrbracket$ implies $z \prec Y\uparrow$ for some finite $Y = \{y_0, \dots, y_n\} \subseteq X$. By (VSINT) there exist $z_0 \prec y_0, \dots, z_n \prec y_n$ such that $z \prec Z\uparrow$, where $Z = \{z_0, \dots, z_n\}$; but then $z_i \in \llbracket y_i \rrbracket$, where $y_i \in X$, so $Z \subseteq I$ and hence $z \in I$. This shows $\llbracket X \rrbracket \subseteq I$ as well.

If $x, y \in D$ and $x \vee y$ exists, we have $\llbracket x \vee y \rrbracket = \{z \in D \mid z \prec x \vee y\} = \{z \in D \mid z \prec \{x, y\}\uparrow\} = \llbracket \{x, y\} \rrbracket = \llbracket x \rrbracket \vee \llbracket y \rrbracket$. ■

Let (D, \leq) be a continuous poset; from now on, we always take the way-below relation \ll as \prec on D . It has been shown in Theorem 2 that the corresponding \prec_{EM} is an auxiliary relation on $(P(D), \leq_{EM})$. We still need

THEOREM 7. *If (D, \leq) is a continuous lattice then \prec_{EM} is a strong auxiliary relation on $(P(D), \leq_{EM})$. In fact, \prec_{EM} has the very strong interpolation property.*

We can now apply Theorem 6 and define the *powerdomain* $P^{CL}(D, \leq)$ of any continuous lattice (D, \leq) as a continuous lattice, namely, the continuous Frink completion $(\|P(D)\|_F, \subseteq)$ of $(P(D), \leq_{EM})$.

The proof of Theorem 7 will require a detailed examination of the structure of upper bounds of finite subsets of $P(D)$. These results will be our main technical tool in the rest of the paper; they are presented next in a series of lemmas.

Let $X_0, \dots, X_n \in P(D)$, where $X_i = \{x_i^0, \dots, x_i^{k_i}\}$, $k_i \geq 0$. We let $\text{Seq} = \{\mathbf{j} = (j_0, \dots, j_n) \mid 0 \leq j_i \leq k_i, 0 \leq i \leq n\}$. For every $\mathbf{j} \in \text{Seq}$ we define

$$x^{\mathbf{j}} = x^{j_0 \dots j_n} = x_0^{j_0} \vee \dots \vee x_n^{j_n}.$$

LEMMA 8. Assume that $S \subseteq \text{Seq}$ has the property

$$(\forall i \leq n) (\forall j \leq k_i) (\exists \mathbf{j} \in S) (\mathbf{j}(i) = j); \quad (*)$$

then $\{x^{\mathbf{j}} \mid \mathbf{j} \in S\} \in \{X_0, \dots, X_n\}^\uparrow$.

Conversely, if $Y \in \{X_0, \dots, X_n\}^\uparrow$ then there is $S \subseteq \text{Seq}$ with the property $(*)$ such that $\{x^{\mathbf{j}} \mid \mathbf{j} \in S\} \leq_{EM} Y$.

The proof of Lemma 8 can be followed with the help of Fig. 1, where $\leq = \supseteq$. There are 7 sets with the property $(*)$, for example, $S = \{(0, 0), (1, 1)\}$, $S' = \{(0, 0), (0, 1), (1, 1)\}$, and $S'' = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Proof. Consider X_i , $i \leq n$. For every $x_i^j \in X_i$ we have some $\mathbf{j} \in S$ such that $\mathbf{j}(i) = j$ and hence $x_i^j \leq x_0^{j_0} \vee \dots \vee x_i^j \vee \dots \vee x_n^{j_n} = x^{\mathbf{j}}$. For every $x^{\mathbf{j}}$, $\mathbf{j} \in S$, we have $x_i^j \leq x^{\mathbf{j}}$, where $j = \mathbf{j}(i)$. This shows $X_i \leq_{EM} \{x^{\mathbf{j}} \mid \mathbf{j} \in S\}$.

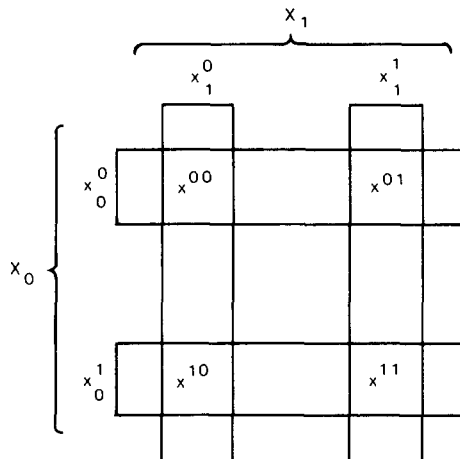


FIGURE 1

Conversely, let $Y \in \{X_0, \dots, X_n\}^\uparrow$; let $S = \{\mathbf{j} \in \text{Seq} \mid (\exists y \in Y) x^{\mathbf{j}} \leq y\}$. Let $y \in Y$; since $X_i \leq_{\text{EM}} Y$, there is some $x_{i'}^j \leq y$. Let $\mathbf{j} = (j_0, \dots, j_n)$; then $x^{\mathbf{j}} = x_0^{j_0} \vee \dots \vee x_n^{j_n} \leq y$ and so $\mathbf{j} \in S$. Thus we see $\{x^{\mathbf{j}} \mid \mathbf{j} \in S\} \leq_{\text{EM}} Y$.

It remains to verify that S has the property (*). If $i \leq n$ and $j \leq k_i$, we have $X_i \leq_{\text{EM}} Y$, $x_i^j \in X_i$, and so there is some $y \in Y$ such that $x_i^j \leq y$. Fix y . For every $i' \neq i$, $X_{i'} \leq_{\text{EM}} Y$ as well, and hence there exists $j_{i'} \leq k_{i'}$ such that $x_{i'}^{j_{i'}} \leq y$. Let $j_i = j$; then $x^{j_0 \dots j_i \dots j_n} = x_0^{j_0} \vee \dots \vee x_i^j \vee \dots \vee x_n^{j_n} \leq y$, so $(j_0, \dots, j_i, \dots, j_n) \in S$. ■

COROLLARY 9. *Every finite subset of $(P(D), \leq_{\text{EM}})$ has a finite complete set of minimal upper bounds.*

An upper bound y of $X \subseteq D$ is called *minimal* if for every $z \in X^\uparrow$, $z \leq y$ implies $z = y$. A set U of upper bounds of X is *complete* if for every $y \in X^\uparrow$ there is some $u \in U$ such that $u \leq y$.

Proof. $\{0\}$ is the least upper bound of \emptyset in $(P(D), \leq_{\text{EM}})$. If $\{X_0, \dots, X_n\} \subseteq P(D)$ is nonempty, Lemma 8 shows that the sets $\{x^{\mathbf{j}} \mid \mathbf{j} \in S\}$, where S ranges over all subsets of Seq with the property (*), form a finite complete set of upper bounds for $\{X_0, \dots, X_n\}$. We take the \leq_{EM} -minimal ones among them to satisfy Corollary 9. ■

LEMMA 10. $X \leq_{\text{EM}} \{X_0, \dots, X_n\}^\uparrow$ if and only if

- (i) $(\forall \mathbf{j} \in \text{Seq}) (\exists x \in X), x \leq x^{\mathbf{j}}$ and
- (ii) $(\forall x \in X) (\exists i \leq n) (\exists j \leq k_i) (x \leq x^{\mathbf{j}} \text{ holds whenever } \mathbf{j}(i) = j)$.

The equivalence also holds if \leq is replaced by \ll and \leq_{EM} by $<_{\text{EM}}$.

Proof. Assume that (i) and (ii) hold. Let $S \subseteq \text{Seq}$ have the property (*). Given $x \in X$, take i and j as in (ii) and use (*) to get $\mathbf{j} \in S$ such that $\mathbf{j}(i) = j$. Then $x \leq x^{\mathbf{j}}$ by (ii). Conversely, given $x^{\mathbf{j}}$, $\mathbf{j} \in S$, there is some $x \in X$ such that $x \leq x^{\mathbf{j}}$, by (i). This shows $X \leq_{\text{EM}} \{x^{\mathbf{j}} \mid \mathbf{j} \in S\}$, for all S with (*). By the proof of the Corollary 9, $X \leq_{\text{EM}} \{X_0, \dots, X_n\}^\uparrow$.

Assume now that (i) and (ii) do not both hold. If (i) fails, there is some $\mathbf{j} \in \text{Seq}$ such that for all $x \in X$, $x \leq x^{\mathbf{j}}$ fails. But then $X \not\leq_{\text{EM}} \{x^{\mathbf{j}} \mid \mathbf{j} \in \text{Seq}\} \in \{X_0, \dots, X_n\}^\uparrow$. (Seq has the property (*)).

If (ii) fails, there is some $x \in X$ such that $(\forall i) (\forall j) (\exists \mathbf{j}) (\mathbf{j}(i) = j \text{ and } x \not\leq x^{\mathbf{j}})$. But this says that $S = \{\mathbf{j} \in \text{Seq} \mid x \not\leq x^{\mathbf{j}}\}$ has the property (*). So we have again $X \not\leq_{\text{EM}} \{x^{\mathbf{j}} \mid \mathbf{j} \in S\} \in \{X_0, \dots, X_n\}^\uparrow$.

Replacing \leq with \ll and \leq_{EM} with $<_{\text{EM}}$ throughout the proof verifies the last claim in Lemma 10. ■

We need one other simple fact.

LEMMA 11. *In any lattice (D, \leq) , if $x_1 = \sup D_1$ and $x_2 = \sup D_2$, where D_1 and D_2 are directed, then $x_1 \vee x_2 = \sup\{y_1 \vee y_2 \mid y_1 \in D_1, y_2 \in D_2\}$. ■*

Proof of Theorem 7. Let us assume $X \prec_{\text{EM}} \{X_0, \dots, X_n\}^\uparrow$, where $X_i = \{x_i^0, \dots, x_i^{k_i}\}$. By Lemma 10, we have

(i) $(\forall \mathbf{j}) (\exists x \in X) x \ll x^{\mathbf{j}}$ and

(ii) $(\forall x \in X) (\exists i) (\exists \mathbf{j}) x \ll x^{\mathbf{j}}$ whenever $\mathbf{j}(i) = j$.

(D, \leq) is a continuous lattice; therefore, for each $i \leq n, j \leq k_n$, we have $x_i^j = \sup\{y \mid y \ll x_i^j\}$. Repeated applications of Lemma 11 show that

$$x^{\mathbf{j}} = x_0^{j_0} \vee \dots \vee x_n^{j_n} = \sup\{y_0^{j_0} \vee \dots \vee y_n^{j_n} \mid y_0^{j_0} \ll x_0^{j_0}, \dots, y_n^{j_n} \ll x_n^{j_n}\}.$$

Consider all sequences $\langle y_i^j \mid i \leq n, j \leq k_i \rangle$ such that $y_i^j \ll x_i^j$ for all $i \leq n, j \leq k_i$; this is a directed system in the pointwise ordering by \leq (because $y \ll x$ and $y' \ll x$ imply $y \vee y' \ll x$ by (P4)). For every pair $(x, x^{\mathbf{j}})$ such that $x \ll x^{\mathbf{j}}$ there is some $\langle y_i^j \mid i \leq n, j \leq k_i \rangle$ such that $x \ll y_0^{j_0} \vee \dots \vee y_n^{j_n} = y^{\mathbf{j}}$, $y_i^j \ll x_i^j$ for all i, j . Since the number of such pairs is finite and the system is directed, we can find a fixed $\langle y_i^j \mid i \leq n, j \leq k_i \rangle$, $y_i^j \ll x_i^j$ for all i, j , such that $x \ll y^{\mathbf{j}}$ whenever $x \ll x^{\mathbf{j}}$. Now let $Y_i = \{y_i^0, \dots, y_i^{k_i}\}$, $i \leq n$; clearly $Y_i \prec_{\text{EM}} X_i$. Since $x \ll x^{\mathbf{j}}$ implies $x \ll y^{\mathbf{j}}$, for all \mathbf{j} , properties (i) and (ii) at the beginning of the proof hold with $x^{\mathbf{j}}$ replaced by $y^{\mathbf{j}}$. Hence $X \prec_{\text{EM}} \{Y_0, \dots, Y_n\}^\uparrow$ by Lemma 10. ■

We record another property of $P(D)$.

LEMMA 12. *Let \mathcal{X} be a finite subset of $P(D)$ and $[\mathcal{X}] = \{X \mid X \leq_{\text{EM}} \mathcal{X}^\uparrow\}$ be the ideal generated by \mathcal{X} . There is a finite set $\mathcal{Z} \subseteq [\mathcal{X}]$ such that $[\mathcal{X}] = \downarrow \mathcal{Z}$.*

It follows that $[\mathcal{X}]$ has a finite complete set of \leq_{EM} -maximal elements.

Proof. Let $\mathcal{X} = \{X_0, \dots, X_n\}$ and $X \leq_{\text{EM}} \mathcal{X}^\uparrow$. Conditions (i) and (ii) from Lemma 10 then hold. For every $x \in X$ let $\tilde{x} = \inf\{x^{\mathbf{j}} \mid x \ll x^{\mathbf{j}}\}$. We note that $x \leq \tilde{x}$ and $\tilde{x} \leq x^{\mathbf{j}}$ hold whenever $x \ll x^{\mathbf{j}}$. Let $\tilde{X} = \{\tilde{x} \mid x \in X\}$; then clearly $X \leq_{\text{EM}} \tilde{X}$ and conditions (i) and (ii) hold with \tilde{X} in place of X , so that $\tilde{X} \leq_{\text{EM}} \mathcal{X}^\uparrow$. We let $\mathcal{Z} = \{\tilde{X} \mid X \in [\mathcal{X}]\}$; \mathcal{Z} is finite because $\{x^{\mathbf{j}} \mid \mathbf{j} \in \text{Seq}\}$ is finite and hence the set of all possible \tilde{x} is finite. ■

THEOREM 13. *Let (D, \leq) be an algebraic lattice. Then there is a unique isomorphism k between $\mathbf{P}^{\text{AL}}(D)$ and $\mathbf{P}^{\text{CL}}(D)$ such that $k(\langle X \rangle) = \langle\langle X \rangle\rangle = \langle X \rangle$ holds for all $X \in P(K(D))$.*

Proof. For any ideal \mathcal{I} of $P(K(D))$ let

$$k(\mathcal{I}) = \{X \in P(D) \mid X \leq_{\text{EM}} \{Y_0, \dots, Y_n\}^\uparrow \text{ for some } Y_0, \dots, Y_n \in \mathcal{I}\}$$

be the least ideal of $P(D)$ containing \mathcal{I} .

It is clear that $k(\mathcal{I})$ is indeed an ideal; if $X \leq_{\text{EM}} \mathcal{Y} \uparrow$ for some finite $\mathcal{Y} \subseteq k(\mathcal{I})$, then $Y \leq_{\text{EM}} \mathcal{Z}_Y \uparrow$ for some finite $Z_Y \subseteq \mathcal{I}$ and each $Y \in \mathcal{Y}$. Let $\mathcal{Z} = \bigcup \{ \mathcal{Z}_Y \mid Y \in \mathcal{Y} \}$; $\mathcal{Z} \subseteq \mathcal{I}$ is finite and $X \leq_{\text{EM}} \mathcal{Z} \uparrow$, so $X \in k(\mathcal{I})$.

$k(\mathcal{I})$ is round. If $X \in P(K(D))$ then clearly $X <_{\text{EM}} X$; therefore $\mathcal{I} \subseteq c(k(\mathcal{I}))$. But $c(k(\mathcal{I})) \subseteq k(\mathcal{I})$ and $k(\mathcal{I})$ is the least ideal containing \mathcal{I} ; hence $k(\mathcal{I}) = c(k(\mathcal{I}))$.

If $\mathcal{I} \subseteq \mathcal{J}$ then $k(\mathcal{I}) \subseteq k(\mathcal{J})$. If $\mathcal{I} \not\subseteq \mathcal{J}$, take $X \in \mathcal{I}$, $X \notin \mathcal{J}$. Then $X \in k(\mathcal{I})$ and $X \notin k(\mathcal{J})$, because the latter would imply $X \leq_{\text{EM}} \mathcal{Y} \uparrow$ for some finite $\mathcal{Y} \subseteq \mathcal{J}$, leading to $X \in \mathcal{J}$. So $k(\mathcal{I}) \not\subseteq k(\mathcal{J})$.

It remains to show that k is onto $\|P(D)\|_F$. Let \mathcal{J} be a round ideal of $P(D)$ and let $\mathcal{I} = \mathcal{J} \cap P(K(D))$. \mathcal{I} is an ideal of $P(K(D))$: if $X \in P(K(D))$ and $X \leq_{\text{EM}} \mathcal{Y} \uparrow$ for $\mathcal{Y} \subseteq \mathcal{I}$ finite, then $\mathcal{Y} \subseteq \mathcal{J}$ and, since \mathcal{J} is an ideal, $X \in \mathcal{J}$, consequently, $X \in \mathcal{I}$. Since $k(\mathcal{I})$ is the least ideal of $P(D)$ containing \mathcal{I} , we get immediately $k(\mathcal{I}) \subseteq \mathcal{J}$. For the converse, let $X \in \mathcal{J}$; since \mathcal{J} is round, we can find $X_0, \dots, X_n \in \mathcal{J}$ such that $X <_{\text{EM}} \{X_0, \dots, X_n\} \uparrow$. By (VSINT), we can further find $Y_i <_{\text{EM}} X_i$, for all $i \leq n$, such that $X <_{\text{EM}} \{Y_0, \dots, Y_n\} \uparrow$. As (D, \leq) is algebraic, we can finally get $Z_i \in P(K(D))$ such that $Y_i \leq_{\text{EM}} Z_i \leq_{\text{EM}} X_i$ (observe that $x \leq y \Leftrightarrow (\exists z \in K(D)) (x \leq z \leq y)$ for $x, y \in D$, and hence $X <_{\text{EM}} Y \Leftrightarrow (\exists Z \in P(K(D))) (X \leq_{\text{EM}} Z \leq_{\text{EM}} Y)$ for $X, Y \in P(D)$). Now $Z_i \in \mathcal{I}$ for all $i \leq n$ and $X \leq_{\text{EM}} \{Z_0, \dots, Z_n\} \uparrow$, showing $X \in k(\mathcal{I})$.

The fact that $k(\langle X \rangle) = \langle\langle X \rangle\rangle = \langle X \rangle$ and the uniqueness of k are obvious. ■

We conclude this section with some further remarks concerning the mapping c . Let (D, \leq) be a poset with a strong auxiliary relation $<$. Theorem 5 shows that $c: |D|_F \rightarrow |D|_F$ is a kernel operator on $|D|_F$ and preserves directed sups. It follows from general properties of kernel operators (Gierz *et al.*, 1980, 0, 3, Proposition 3.12) that the corestriction $c: |D|_F \rightarrow \|D\|_F$ preserves arbitrary infs. Less trivial is

LEMMA 14. *If $<$ has the very strong interpolation property then $c: |D|_F \rightarrow |D|_F$ preserves arbitrary sups.*

Proof. It remains to prove preservation of finite sups. Let $I, J \in |D|_F$; the nontrivial direction is $c(I \vee J) \subseteq c(I) \vee c(J)$. If $x \in c(I \vee J)$ then $x < Z \uparrow$ for some finite $Z \subseteq I \vee J = \{z \mid z \leq (X \cup Y) \uparrow \text{ for some finite } X \subseteq I, Y \subseteq J\}$. Hence one can find finite $X \subseteq I$, $Y \subseteq J$ such that $x < (X \cup Y) \uparrow$. Let $X = \{x_0, \dots, x_n\}$, $Y = \{y_0, \dots, y_m\}$; by (VSINT) we can find $X' = \{x'_0, \dots, x'_n\}$ and $Y' = \{y'_0, \dots, y'_m\}$ so that $x < (X' \cup Y') \uparrow$ and $x'_i < x_i$, $y'_j < y_j$ hold for all i, j . Consequently $X' \subseteq c(I)$, $Y' \subseteq c(J)$, and $x \in c(I) \vee c(J)$. ■

This result will be useful later; here we only point out that it suggests an alternative (dual) approach to the construction of $P^{\text{CL}}(D)$. By Gierz *et al.*

(1980, 0, 3), (see also Section 3), the kernel operator $c: |D|_F \rightarrow |D|_F$ has an upper adjoint $\check{c}: |D|_F \rightarrow |D|_F$ defined by $\check{c}(I) = \sup\{J \mid c(J) \subseteq I\}$. It is easy to check that \check{c} is a *closure operator* and its range $\|D\|_F$ consists exactly of all “pointed” ideals, where we call $I \in |D|_F$ *pointed* if $\downarrow\{x\} \subseteq I$ implies $x \in I$. The restricted maps $\check{c}: \|D\|_F \rightarrow \|D\|_F$ and $c: \|D\|_F \rightarrow \|D\|_F$ are isomorphisms. These observations applied to $(P(D), <_{EM})$ show that $(\|P(D)\|_F, \subseteq)$ is isomorphic to $P^{CL}(D)$. We could thus use pointed ideals in place of round ideals; it would have some advantages (e.g., $\check{c}([X]) = [X]$ for finite X , while $c([X]) = \llbracket X \rrbracket \subsetneq [X]$, in general) and some disadvantages (e.g., sup of a directed set of pointed ideals is in general not its union), and we shall use only round ideals in the rest of the paper.

3. CONTINUOUS FUNCTIONS

Domains required by denotational semantics are often obtained as solutions of *domain equations*, i.e., equations of the form $D \cong T(D)$, where T is some functor in the category of domains. We want to be able to use the power domain construct P in such equations; for this reason, the present section is devoted to the study of the functorial properties of P .

We first consider the (from the point of view of denotational semantics) less interesting categories **AP** and **CP**. Let $f \in [D \rightarrow E]$, where D and E are continuous cpo's. f naturally extends to a mapping of $P(D)$ into $P(E)$: we let $f(X) = f[X] = \{f(x) \mid x \in X\} \in P(E)$ for every $X \in P(D)$. Notice that $X \leq_{EM} Y$ implies $f(X) \leq_{EM} f(Y)$, so $f: P(D) \rightarrow P(E)$ is well defined (i.e., $X \equiv_{EM} Y \Rightarrow f(X) \equiv_{EM} f(Y)$) and monotone. Next, we define $Pf: |P(D)| \rightarrow |P(E)|$ so that, for any directed ideal \mathcal{J} of $P(D)$,

$$Pf(\mathcal{J}) = \{Y \in P(E) \mid Y \leq_{EM} f(X) \text{ for some } X \in \mathcal{J}\}.$$

It is clear that $Pf(\mathcal{J})$ is a directed ideal of $P(E)$ and Pf is continuous (as a mapping of $|P(D)|$ into $|P(E)|$). Also $P1_D = 1_D$, where $1_D \in [D \rightarrow D]$ is the identity on D , and $P(g \circ f) = Pg \circ Pf$ whenever $f \in [D \rightarrow E]$ and $g \in [E \rightarrow F]$.

Finally, for $\mathcal{J} \in \|P(D)\|$ we let $P^{CP}f(\mathcal{J}) = c(Pf(\mathcal{J})) = \{Y \in P(E) \mid Y <_{EM} f(X) \text{ for some } X \in \mathcal{J}\} \in \|P(E)\|$. By Proposition 4 in Section 1, c maps $|P(E)|$ continuously into $\|P(E)\|$, and hence $P^{CP}f$ is a continuous mapping of $\|P(D)\|$ into $\|P(E)\|$.

LEMMA 1. $c(Pf(c(\mathcal{J}))) = c(Pf(\mathcal{J}))$ for all $\mathcal{J} \in |P(D)|$.

Proof. This is a special case of Lemma 7 proved later. ▀

THEOREM 2. P^{CP} is a functor in the category of continuous posets, and P^{AP} is a functor in the category of algebraic posets.

Proof. Clearly $P^{CP}1_D = 1_D$. If $f \in [D \rightarrow E]$ and $g \in [E \rightarrow F]$ then $P^{CP}g \circ P^{CP}f(\mathcal{J}) = c(Pg(c(Pf(\mathcal{J})))) = c(Pg(Pf(\mathcal{J}))) = c(P(g \circ f)(\mathcal{J})) = P^{CP}(g \circ f)(\mathcal{J})$. (The second equality is justified by Lemma 1.)

The result for algebraic posets follows immediately from Theorem 4 in Section 2; of course, we define $P^{AP}f$ for $f \in [D \rightarrow E]$, D, E algebraic, so as to make the following diagram commute:

$$\begin{array}{ccc} P^{CP}(D) & \xrightarrow{P^{CP}f} & P^{CP}(E) \\ k_D \uparrow & & \uparrow k_E \\ P^{AP}(D) & \xrightarrow{P^{AP}f} & P^{AP}(E) \end{array}$$

($P^{AP}f = k_E^{-1} \circ P^{CP}f \circ k_D$). In fact, we then have $P^{AP}f(\mathcal{J}) = \{Y \in P(K(E)) \mid Y \leq_{EM} f(X) \text{ for some } X \in \mathcal{J}\}$. (With this definition, k establishes a natural isomorphism between the functor P^{AP} and the restriction of the functor P^{CP} to the category AP .) ■

Another consequence of Lemma 1 provides further insight into the nature of $P^{CP}f$.

THEOREM 3. *Let D, E be continuous posets, $f \in [D \rightarrow E]$. Then $P^{CP}f$ is the unique $F \in [P^{CP}(D) \rightarrow P^{CP}(E)]$ such that $F(\langle\langle X \rangle\rangle) = \langle\langle f(X) \rangle\rangle$ for all $X \in P(D)$.*

Proof. $P^{CP}f(\langle\langle X \rangle\rangle) = c(Pf(c(\langle X \rangle))) = c(Pf(\langle X \rangle)) = c(\langle f(X) \rangle) = \langle\langle f(X) \rangle\rangle$.

By Theorem 3 in Section 2, for any $\mathcal{J} \in \|P(D)\|$, $\mathcal{J} = \sup\{\langle\langle X \rangle\rangle \mid X \in \mathcal{J}\}$, so any continuous function on $P^{CP}(D)$ is determined by its values on $\langle\langle X \rangle\rangle$, $X \in P(D)$. ■

Let us now turn to continuous functions on continuous and algebraic lattices. Here the situation is more complicated because Frink completions of $(P(D), \leq_{EM})$ introduce new finite sups; with continuity as the only requirement, there need not be a *unique* way of extending $f: P(D) \rightarrow P(E)$ to all of $|P(D)|_F$ or $\|P(D)\|_F$.

Let D, E be continuous lattices and let $f \in [D \rightarrow E]$. For all $\mathcal{J} \in |P(D)|_F$ we define

$$Pf(\mathcal{J}) = \{Y \in P(E) \mid Y \leq_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow, \text{ where } X_0, \dots, X_n \in \mathcal{J}\}$$

and

$$\bar{P}f(\mathcal{J}) = \{Y \in P(E) \mid Y \leq_{EM} f[\{X_0, \dots, X_n\}^\uparrow], \text{ where } X_0, \dots, X_n \in \mathcal{J}\}.$$

Here and elsewhere, $f[\{X_0, \dots, X_n\}^\uparrow] = \{f(Z) \mid Z \in \{X_0, \dots, X_n\}^\uparrow\}$.

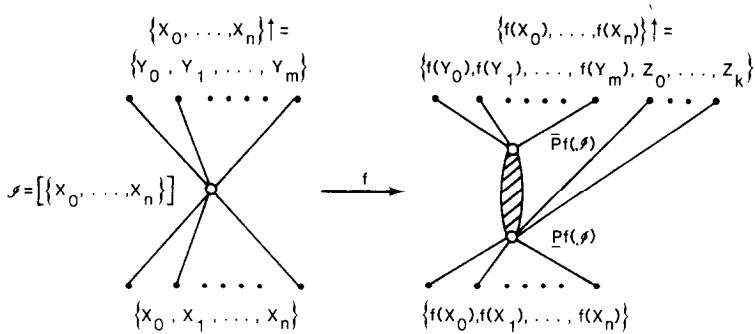


FIGURE 2

Figure 2 illustrates the situation in general terms. A more detailed picture of how $\underline{P}f(\mathcal{J}) \subsetneq \bar{P}f(\mathcal{J})$ can occur (easily converted into a formal example) is given in Fig. 3.

LEMMA 4. (i) $\underline{P}f$ and $\bar{P}f$ are continuous mappings of $|P(D)|_F$ into $|P(E)|_F$ and $\underline{P}f(\langle X \rangle) = \bar{P}f(\langle X \rangle) = \langle f(X) \rangle$ for all $X \in P(D)$.

(ii) $\underline{P}f \leq \bar{P}f$.

(iii) If $P \in [|P(D)|_F \rightarrow |P(E)|_F]$ is such that $P(\langle X \rangle) = \langle f(X) \rangle$ for all $X \in P(D)$, then $\underline{P}f \leq P \leq \bar{P}f$.

(iv) $\underline{P}(g \circ f) \leq \underline{P}g \circ \underline{P}f$; $\bar{P}(g \circ f) \geq \bar{P}g \circ \bar{P}f$.

(v) If $f \leq g$ then $\underline{P}f \leq \underline{P}g$ and $\bar{P}f \leq \bar{P}g$.

Proof. (i) $\underline{P}f(\mathcal{J})$ is an ideal: if $Y \leq_{EM} \{Y_0, \dots, Y_k\}^\uparrow$ for $Y_0, \dots, Y_k \in \underline{P}f(\mathcal{J})$, then $Y_i \leq_{EM} \{f(X_0^i), \dots, f(X_{n_i}^i)\}^\uparrow$, where $X_0^i, \dots, X_{n_i}^i \in \mathcal{J}$. Letting $\{X_0, \dots, X_n\} = \{X_0^0, \dots, X_{n_0}^0, \dots, X_0^k, \dots, X_{n_k}^k\} \subseteq \mathcal{J}$ we have $Y \leq_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow$ and hence $Y \in \underline{P}f(\mathcal{J})$. In the same way one shows $\bar{P}f(\mathcal{J})$ to be an ideal. The rest of (i) is straightforward.

(ii) follows from (i) and (iii).

(iii) If $Y \in \underline{P}f(\mathcal{J})$ then $Y \leq_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow$, where $X_0, \dots, X_n \in \mathcal{J}$. For every $i \leq n$, $X_i \in \mathcal{J}$, so $\langle X_i \rangle \subseteq \mathcal{J}$ and $\langle f(X_i) \rangle = P(\langle X_i \rangle) \subseteq P(\mathcal{J})$, giving $f(X_i) \in P(\mathcal{J})$. Since $P(\mathcal{J})$ is an ideal, we get $Y \in P(\mathcal{J})$. Thus $\underline{P}f(\mathcal{J}) \subseteq P(\mathcal{J})$.

Let now $Y \in P(\mathcal{J})$; by continuity of P , $Y \in P([\{X_0, \dots, X_n\}])$ for some $X_0, \dots, X_n \in \mathcal{J}$ (Theorem 1 in Section 2). If $Z \in \{X_0, \dots, X_n\}^\uparrow$ then $[\{X_0, \dots, X_n\}] \subseteq \langle Z \rangle$, so $P([\{X_0, \dots, X_n\}]) \subseteq P(\langle Z \rangle) = \langle f(Z) \rangle$ and $Y \in \langle f(Z) \rangle$, i.e., $Y \leq_{EM} f(Z)$. This shows $Y \in \bar{P}f(\mathcal{J})$.

(iv) $P = \underline{P}g \circ \underline{P}f$ is continuous and $P(\langle X \rangle) = \underline{P}g(\underline{P}f(\langle X \rangle)) = \langle g(f(X)) \rangle = \langle g \circ f(X) \rangle$ for all $X \in P(D)$, by (i). Applying (iii) yields $\underline{P}(g \circ f) \leq P \leq \bar{P}(g \circ f)$. The same argument works for $\bar{P}g \circ \bar{P}f$.

(v) Let $f \leq g$; for any $X_0, \dots, X_n \in \mathcal{J}$ we have $f(X_0) \leq_{\text{EM}} g(X_0), \dots, f(X_n) \leq_{\text{EM}} g(X_n)$, hence $\{f(X_0), \dots, f(X_n)\} \uparrow \supseteq \{g(X_0), \dots, g(X_n)\}$ and $\underline{P}f(\mathcal{J}) \subseteq \underline{P}g(\mathcal{J})$. The proof for $\bar{P}f$ is similar. ■

Finally, we define $\underline{P}^{\text{CL}}f = (c \circ \underline{P}f) \upharpoonright \|P(D)\|_F$ and $\bar{P}^{\text{CL}}f = (c \circ \bar{P}f) \upharpoonright \|P(D)\|_F$. We have to establish the basic properties of these operations. Figure 3 should be of some help in following the proofs of the next two rather technical, but important, results.

LEMMA 5. *If $X <_{\text{EM}} \{f(X_0), \dots, f(X_n)\} \uparrow$ then there exist $Y_0 <_{\text{EM}} X_0, \dots, Y_n <_{\text{EM}} X_n$ such that $X <_{\text{EM}} \{f(Y_0), \dots, f(Y_n)\} \uparrow$.*

Proof. The proof is similar to that of Theorem 7 in Chapter 2. We use the notational conventions preceding Lemma 8 in Section 2. We also let $f(x)^j = f(x_0^j) \vee \dots \vee f(x_n^j)$. Note that $f(x)^j \leq f(x^j)$. By Lemma 10 in Section 2, $X <_{\text{EM}} \{f(X_0), \dots, f(X_n)\} \uparrow$ means that

- (i) $(\forall j \in \text{Seq}) (\exists x \in X) x \leq f(x)^j$ and
- (ii) $(\forall x \in X) (\exists i \leq n) (\exists j \leq k_i) x \leq f(x)^j$ holds whenever $j(i) = j$.

Consider the set \mathcal{S} of all sequences $\langle y_i^j \mid i \leq n, j \leq k_i \rangle$ such that $y_i^j \leq x_i^j$ for all $i \leq n, j \leq k_i$; \mathcal{S} is a directed system in the pointwise ordering by \leq . Moreover, $f(x_i^j) = \sup\{f(y_i^j) \mid \langle y_i^j \rangle \in \mathcal{S}\}$, by continuity of f . Lemma 11 in Section 2 now shows that $f(x)^j = \sup\{f(y)^j \mid \langle y_i^j \rangle \in \mathcal{S}\}$ holds for each $j \in \text{Seq}$. Using this and the fact that \mathcal{S} is directed, we find $\langle y_i^j \rangle \in \mathcal{S}$ such that, for any $x \in X$ and $j \in \text{Seq}$, if $x \leq f(x)^j$ then $x \leq f(y)^j$. Now (i) and (ii)

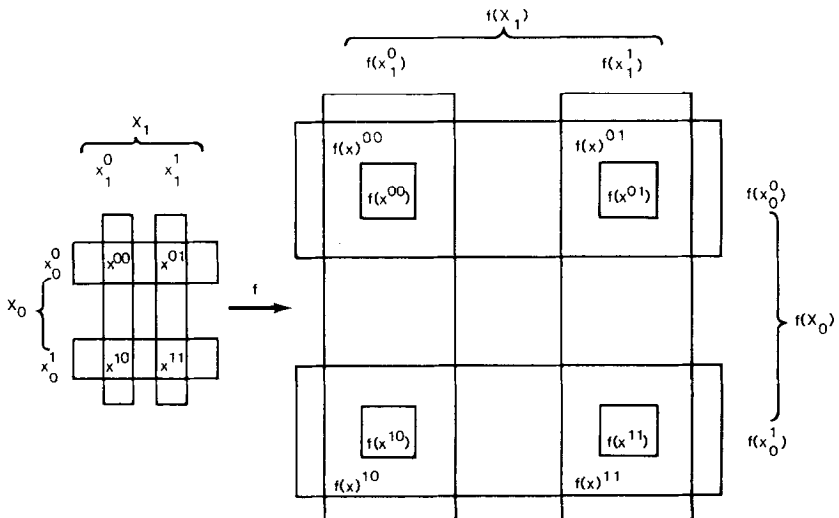


FIGURE 3

above hold with $f(x)^j$ replaced by $f(y)^j$, so by Lemma 10 again, $X <_{\text{EM}} \{f(Y_0), \dots, f(Y_n)\}^\uparrow$, where $Y_i = \{y_i^j \mid j \leq k_i\}$ for all $i \leq n$. ■

LEMMA 6. *If $X <_{\text{EM}} f[\{X_0, \dots, X_n\}^\uparrow]$ then there exist $Y_0 <_{\text{EM}} X_0, \dots, Y_n <_{\text{EM}} X_n$ such that $X <_{\text{EM}} f[\{Y_0, \dots, Y_n\}^\uparrow]$.*

Proof. By mimicking the *proof* of Lemma 10 in Section 2 we can show the following fact: $X <_{\text{EM}} f[\{X_0, \dots, X_n\}^\uparrow]$ if and only if

- (i) $(\forall j \in \text{Seq}) (\exists x \in X) x \leq f(x^j)$ and
- (ii) $(\forall x \in X) (\exists i \leq n) (\exists j \leq k_i) x \leq f(x^j)$ holds whenever $j(i) = j$.

We now proceed just as in the proof of Lemma 5 but using the above in place of Lemma 10. (In particular, note that $x^j = \sup\{y^j \mid \langle y^j \rangle \in \mathcal{S}\}$ and hence $f(x^j) = \sup\{f(y^j) \mid \langle y^j \rangle \in \mathcal{S}\}$.)

LEMMA 7. *For all $\mathcal{S} \in |P(D)|_F$,*

- (i) $c(\underline{P}f(c(\mathcal{S}))) = c(\underline{P}f(\mathcal{S}))$;
- (ii) $c(\bar{P}f(c(\mathcal{S}))) = c(\bar{P}f(\mathcal{S}))$.

Proof. \subseteq is trivial from $c(\mathcal{S}) \subseteq \mathcal{S}$ and monotonicity of $\underline{P}f$, $\bar{P}f$, and c . We have to establish \supseteq . We shall only prove (ii); (i) is similar, but easier.

Let $Y \in c(\bar{P}f(\mathcal{S}))$; then $Y <_{\text{EM}} \{Y_0, \dots, Y_n\}^\uparrow$ for $Y_0, \dots, Y_n \in \bar{P}f(\mathcal{S})$, and $Y_i \leq_{\text{EM}} f[\{X_0^i, \dots, X_{k_i}^i\}^\uparrow]$ for $X_0^i, \dots, X_{k_i}^i \in \mathcal{S}$. We let $\{X_0, \dots, X_m\} = \bigcup \{\{X_0^i, \dots, X_{k_i}^i\} \mid i \leq n\}$; then $Y_i \leq_{\text{EM}} f[\{X_0, \dots, X_m\}^\uparrow]$ holds for all $i \leq n$. By (VSINT) there exist $Z_i <_{\text{EM}} Y_i$ such that $Y <_{\text{EM}} \{Z_0, \dots, Z_n\}^\uparrow$. Using Lemma 6 for each $i \leq n$ and the fact that $\{U_i \mid U_i <_{\text{EM}} X_i\}$ is directed, we can find $U_0 <_{\text{EM}} X_0, \dots, U_m <_{\text{EM}} X_m$ so that $Z_i <_{\text{EM}} f[\{U_0, \dots, U_m\}^\uparrow]$, all $i \leq n$. Now we have $U_i \in c(\mathcal{S})$, $Z_i \in \bar{P}f(c(\mathcal{S}))$, and finally $Y \in c(\bar{P}f(c(\mathcal{S})))$, as needed. ■

THEOREM 8. *Let $f \in [D \rightarrow E]$.*

(o) $\underline{P}^{\text{CL}}f(\mathcal{S}) = \{Y \in P(E) \mid Y <_{\text{EM}} \{f(X_0), \dots, f(X_n)\}^\uparrow, \text{ where } X_0, \dots, X_n \in \mathcal{S}\}$; $\bar{P}^{\text{CL}}f(\mathcal{S}) \subseteq \{Y \in P(E) \mid Y <_{\text{EM}} f[\{X_0, \dots, X_n\}^\uparrow], \text{ where } X_0, \dots, X_n \in \mathcal{S}\}$.

(i) $\underline{P}^{\text{CL}}f$ and $\bar{P}^{\text{CL}}f$ are continuous mappings of $\|P(D)\|_F$ into $\|P(E)\|_F$ and $\underline{P}^{\text{CL}}f(\mathcal{S}) = \bar{P}^{\text{CL}}f(\mathcal{S}) = P^{\text{CP}}f(\mathcal{S})$ if \mathcal{S} is directed.

(ii) $\underline{P}^{\text{CL}}f \leq \bar{P}^{\text{CL}}f$.

(iii) If $P \in [\|P(D)\|_F \rightarrow \|P(E)\|_F]$ is such that $P(\ll X \gg) = \ll f(X) \gg$ for all $X \in P(D)$, then $\underline{P}^{\text{CL}}f \leq P \leq \bar{P}^{\text{CL}}f$.

(iv) $\underline{P}^{\text{CL}}(g \circ f) \leq \underline{P}^{\text{CL}}g \circ \underline{P}^{\text{CL}}f$; $\bar{P}^{\text{CL}}(g \circ f) \geq \bar{P}^{\text{CL}}g \circ \bar{P}^{\text{CL}}f$.

(v) If $f \leq g$ then $\underline{P}^{\text{CL}}f \leq \underline{P}^{\text{CL}}g$ and $\bar{P}^{\text{CL}}f \leq \bar{P}^{\text{CL}}g$.

Proof. (o) If $Y \in \underline{P}^{CL}f(\mathcal{J})$ then $Y <_{EM} \{Y_0, \dots, Y_k\} \uparrow$, where $Y_i \leq_{EM} \{f(X_0^i), \dots, f(X_{n_i}^i)\} \uparrow$, $X_0^i, \dots, X_{n_i}^i \in \mathcal{J}$. Let $\{X_0, \dots, X_n\} = \{X_0^0, \dots, X_{n_0}^0, \dots, X_0^k, \dots, X_{n_k}^k\}$; then $X_0, \dots, X_n \in \mathcal{J}$ and $Y <_{EM} \{f(X_0), \dots, f(X_n)\} \uparrow$. Similarly for \underline{P}^{CL} .

Conversely, if $Y <_{EM} \{f(X_0), \dots, f(X_n)\} \uparrow$, where $X_0, \dots, X_n \in \mathcal{J}$ then $f(X_0), \dots, f(X_n) \in \underline{P}f(\mathcal{J})$ and $Y \in c(\underline{P}f(\mathcal{J})) = \underline{P}^{CL}f(\mathcal{J})$.

(i) and (ii) mostly follow immediately from Lemma 4 and the properties of c listed in Theorem 5 of Section 2; the verification of $\underline{P}^{CL}f(\mathcal{J}) = \bar{P}^{CL}f(\mathcal{J}) = P^{CP}f(\mathcal{J})$ is easy.

(iii) If $X \in \underline{P}^{CL}f(\mathcal{J})$ then $X <_{EM} \{f(X_0), \dots, f(X_n)\} \uparrow$ for $X_0, \dots, X_n \in \mathcal{J}$ (part (o) of this theorem). By (VSINT) there exist $Y_0 <_{EM} f(X_0), \dots, Y_n <_{EM} f(X_n)$ such that $X <_{EM} \{Y_0, \dots, Y_n\} \uparrow$. For each $i \leq n$, $X_i \in \mathcal{J}$, so $\langle\langle X_i \rangle\rangle \subseteq \mathcal{J}$ and $\langle\langle f(x_i) \rangle\rangle = P(\langle\langle X_i \rangle\rangle) \subseteq P(\mathcal{J})$. So $Y_i \in P(\mathcal{J})$ for all $i \leq n$ and $X \in P(\mathcal{J})$, since $P(\mathcal{J})$ is an ideal. This shows $\underline{P}^{CL}f(\mathcal{J}) \subseteq P(\mathcal{J})$. It remains to prove $P(\mathcal{J}) \subseteq \bar{P}^{CL}f(\mathcal{J})$. By Theorem 6 of Section 2 (and continuity of P and $\bar{P}^{CL}f$) it is enough to prove it for $\mathcal{J} = \llbracket \{X_0, \dots, X_n\} \rrbracket$, where $X_0, \dots, X_n \in \mathcal{J}$. For all $Z \in \{X_0, \dots, X_n\} \uparrow$, $\mathcal{J} \subseteq \langle\langle Z \rangle\rangle$ and hence $P(\mathcal{J}) \subseteq P(\langle\langle Z \rangle\rangle) = \langle\langle f(Z) \rangle\rangle$. In other words, $P(\mathcal{J}) \subseteq \{Y \in P(E) \mid Y <_{EM} f(Z) \text{ for all } Z \in \{X_0, \dots, X_n\} \uparrow\} \subseteq \bar{P}f(\mathcal{J})$. Since $P(\mathcal{J})$ is round, $P(\mathcal{J}) \subseteq c(\bar{P}f(\mathcal{J})) = \bar{P}^{CL}f(\mathcal{J})$.

(iv) Using Lemma 7 and Lemma 4(iv), $\underline{P}^{CL}g \circ \underline{P}^{CL}f(\mathcal{J}) = c(Pg(c(\underline{P}f(\mathcal{J})))) = c(Pg(\underline{P}f(\mathcal{J}))) \supseteq c(P(g \circ f)(\mathcal{J})) = \underline{P}^{CL}(g \circ f)(\mathcal{J})$; the proof for \bar{P}^{CL} is analogous.

(v) Apply c to Lemma 4(v). ■

If D and E are algebraic lattices and $f \in [D \rightarrow E]$, we again define $\underline{P}^{AL}f$ and $\bar{P}^{AL}f$ so as to make a diagram like that at the end of the proof of Theorem 2 commute. Hence for $\mathcal{J} \in |P(K(D))|_F$ we have

$$\underline{P}^{AL}f(\mathcal{J}) = \underline{P}^{CL}f(k_D(\mathcal{J})) \cap P(K(E))$$

and

$$\bar{P}^{AL}f(\mathcal{J}) = \bar{P}^{CL}f(k_D(\mathcal{J})) \cap P(K(E))$$

(see the proof of Theorem 13 in Section 2 for the justification of $k_E^{-1}(\mathcal{J}) = \mathcal{J} \cap P(K(E))$.) There is a simpler explicit description of these operators (see Hrbacek, 1985):

CLAIM. For $\mathcal{J} \in |P(K(D))|_F$,

(a) $\underline{P}^{AL}f(\mathcal{J}) = \{Y \in P(K(E)) \mid Y \leq_{EM} \{f(X_0), \dots, f(X_n)\} \uparrow, \text{ where } X_0, \dots, X_n \in \mathcal{J}\}$ and

(b) $\bar{P}^{AL}f(\mathcal{J}) = \{Y \in P(K(E)) \mid Y \leq_{EM} f[\{X_0, \dots, X_n\} \uparrow], \text{ where } X_0, \dots, X_n \in \mathcal{J}\}$.

Proof. (a) By definition and Theorem 8(o), $P^{AL}f(\mathcal{J}) = \{Y \in P(K(E)) \mid Y \prec_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow, \text{ where } X_0, \dots, X_n \in k_D(\mathcal{J})\}$. Since $\mathcal{J} \subseteq k_D(\mathcal{J})$ and $Y \prec_{EM} Y$ holds for $Y \in P(K(E))$, we immediately have \supseteq . For the converse, we start with $Y \in P(K(E))$, $Y \prec_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow$ for $X_i \in k_D(\mathcal{J})$, $i \leq n$, and use Lemma 5 to get $Z_i \prec_{EM} X_i$ such that $Y \prec_{EM} \{f(Z_0), \dots, f(Z_n)\}^\uparrow$. Using the observation at the end of the proof of Theorem 13 in Section 2, we get $T_i \in P(K(D))$ such that $Z_i \leq_{EM} T_i \leq X_i$. So now $T_i \in k_D(\mathcal{J}) \cap P(K(D)) = \mathcal{J}$, $i \leq n$, and $Y \leq_{EM} \{f(T_0), \dots, f(T_n)\}^\uparrow$, proving \subseteq .

(b) An analogous argument, using Lemma 6 in place of Lemma 5, shows that $\{Y \in P(K(E)) \mid Y \prec_{EM} f[\{X_0, \dots, X_n\}^\uparrow], \text{ where } X_0, \dots, X_n \in k_D(\mathcal{J})\} = \{Y \in P(K(E)) \mid Y \leq_{EM} f[\{X_0, \dots, X_n\}^\uparrow], \text{ where } X_0, \dots, X_n \in \mathcal{J}\}$. \subseteq in (b) now follows from Theorem 8(o), while \supseteq is trivial from $Y \prec_{EM} Y$. ■

The equality $\underline{P}^{CL}f = \bar{P}^{CL}f$ does not hold, in general, for arbitrary continuous functions. I do not know whether it is possible to define P so that Pf is continuous for arbitrary continuous f , $\underline{P}^{CL}f \leq Pf \leq \bar{P}^{CL}f$, and $P(g \circ f) = Pg \circ Pf$, and so make the power domain constructor into a functor in the category **CONT** (or **ALG**). However, we show in the next lemma that the equality $\underline{P}^{CL}f = \bar{P}^{CL}f$ does hold for large classes of continuous functions; if it does, we write simply $P^{CL}f$ for either of the above.

LEMMA 9. *Let $f \in [D \rightarrow E]$; if f preserves finite sups or finite infs then $\underline{P}^{CL}f = \bar{P}^{CL}f$.*

Proof. (1) Assume f preserves finite sups. By Lemma 10 in Section 2 we have $X \leq_{EM} \{f(X_0), \dots, f(X_n)\}^\uparrow$ if and only if

- (i) $(\forall \mathbf{j} \in \text{Seq}) (\exists x \in X) x \leq f(x)^\mathbf{j}$ and
- (ii) $(\forall x \in X) (\exists i \leq n) (\exists j \leq k_i) x \leq f(x)^\mathbf{j}$ whenever $\mathbf{j}(i) = j$.

Similarly $X \leq_{EM} f[\{X_0, \dots, X_n\}^\uparrow]$ if and only if (i) and (ii) hold with $f(x)^\mathbf{j}$ in place of $f(x)^\mathbf{j}$ (see the proof of Lemma 6). But, if f preserves finite sups, then $f(x^\mathbf{j}) = f(x_0^{j_0} \vee \dots \vee x_n^{j_n}) = f(x_0^{j_0}) \vee \dots \vee f(x_n^{j_n}) = f(x)^\mathbf{j}$, so we have $\underline{P}f(\mathcal{J}) = \bar{P}f(\mathcal{J})$ and hence $\underline{P}^{CL}f(\mathcal{J}) = \bar{P}^{CL}f(\mathcal{J})$.

We remark that this argument works with \prec_{EM} in place of \leq_{EM} as well, proving equality in the second equation of Theorem 8(o) for such f .

(2) Assume f preserves finite infs. Let $X_0, \dots, X_n \in \mathcal{J}$ and $Y \prec_{EM} f[\{X_0, \dots, X_n\}^\uparrow]$. As in the proof of Lemma 6, we have

- (i) $(\forall \mathbf{j} \in \text{Seq}) (\exists y \in Y) y \ll f(x)^\mathbf{j}$ and
- (ii) $(\forall y \in Y) (\exists i \leq n) (\exists j \leq k_i) y \ll f(x)^\mathbf{j}$ whenever $\mathbf{j}(i) = j$.

For every $y \in Y$ we let $x_y = \inf\{x^\mathbf{j} \mid y \ll f(x)^\mathbf{j}\}$; finally we let $X = \{x_y \mid y \in Y\}$.

With these definitions, whenever $y \ll f(x^j)$, we have $x_y \leq x^j$; so

$$(i') \quad (\forall j \in \text{Seq}) \quad (\exists x \in X) \quad (x \leq x^j) \text{ and}$$

$$(ii') \quad (\forall x \in X) \quad (\exists i \leq n) \quad (\exists j \leq k_i) \quad x \leq x^j \text{ whenever } j(i) = j$$

both hold. By Lemma 10 in Section 2 we get $X \leq_{\text{EM}} \{X_0, \dots, X_n\}^\uparrow$ and hence $X \in \mathcal{J}$. Now, f preserves finite infs, so $f(x_y) = \inf\{f(x^j) \mid y \ll f(x^j)\}$ and hence $y \leq f(x_y)$ for each $y \in Y$. It follows that $Y \leq_{\text{EM}} f(X)$; since $X \in \mathcal{J}$, $Y \in \underline{Pf}(\mathcal{J})$. Thus we have proved that $\{Y \in P(E) \mid Y <_{\text{EM}} f[\{X_0, \dots, X_n\}^\uparrow]\}$, where $X_0, \dots, X_n \in \mathcal{J} \subseteq \underline{Pf}(\mathcal{J})$; by applying c to both sides (and using Theorem 8(o)) we get $\bar{\mathbf{P}}^{\text{CL}}f(\mathcal{J}) \subseteq \underline{\mathbf{P}}^{\text{CL}}f(\mathcal{J})$. ■

We note that (2) actually shows: If $f \in [D \rightarrow E]$ preserves finite infs then $\underline{\mathbf{P}}^{\text{CL}}f(\mathcal{J}) = \bar{\mathbf{P}}^{\text{CL}}f(\mathcal{J}) \subseteq \{Y \in P(E) \mid Y \leq_{\text{EM}} f(X) \text{ for some } X \in \mathcal{J}\}$; precisely $\underline{\mathbf{P}}^{\text{CL}}f(\mathcal{J}) = \bar{\mathbf{P}}^{\text{CL}}f(\mathcal{J}) = c(\{Y \in P(E) \mid Y \leq_{\text{EM}} f(X) \text{ for some } X \in \mathcal{J}\})$. (Compare with Lemma 12 in Section 2.)

We shall now consider the category **CL** of all continuous lattices and all continuous inf-preserving maps, and its full subcategory **AL** of algebraic lattices.

THEOREM 10. \mathbf{P}^{CL} is a functor in the category **CL** and \mathbf{P}^{AL} is a functor in the category **AL**.

Proof. $\mathbf{P}^{\text{CL}}1_D = 1_{\mathbf{P}^{\text{CL}}(D)}$ is trivial and $\mathbf{P}^{\text{CL}}(g \circ f) = \mathbf{P}^{\text{CL}}g \circ \mathbf{P}^{\text{CL}}f$ follows from Theorem 8(iv) and Lemma 9. So we only need to prove

CLAIM. If $f \in [D \rightarrow E]$ preserves infs then $\mathbf{P}^{\text{CL}}f \in [\|P(D)\|_F \rightarrow \|P(E)\|_F]$ preserves infs.

Proof of Claim. Let $\mathcal{J} = c(\mathcal{J})$, where $\mathcal{J} = \bigcap \{\mathcal{J}_i \mid i \in I\}$ (this is how the infimum of a set of round ideals in $\|P(D)\|_F$ is computed). $\mathbf{P}^{\text{CL}}f$ is monotone and $\mathbf{P}^{\text{CL}}f(\mathcal{J})$ is round, so certainly $\mathbf{P}^{\text{CL}}f(\mathcal{J}) \subseteq c(\bigcap \{\mathbf{P}^{\text{CL}}f(\mathcal{J}_i) \mid i \in I\})$. Now assume that $Y \in \mathbf{P}^{\text{CL}}f(\mathcal{J}_i)$ for each $i \in I$. By the remark following the proof of Lemma 9, there is some $X_i \in \mathcal{J}_i$ such that $Y \leq_{\text{EM}} f(X_i)$. We let $x_y = \inf\{x \mid x \in \bigcup \{X_i \mid i \in I\} \text{ and } y \leq f(x)\}$ and note that for every $y \in Y$ there is some $x \in X_i$ such that $x_y \leq x$. Since f preserves arbitrary infs, we have $f(x_y) = \inf\{f(x) \mid x \in \bigcup \{X_i \mid i \in I\} \text{ and } y \leq f(x)\}$ and $y \leq f(x_y)$. Finally let $X = \{x_y \mid y \in Y\}$. Then $Y \leq_{\text{EM}} f(X)$ and $X \leq_{\text{EM}} X_i$ for each $i \in I$, i.e., $X \in \bigcap \{\mathcal{J}_i \mid i \in I\} = \mathcal{J}$ and $Y \in \underline{Pf}(\mathcal{J})$.

Thus we have shown that

$$\bigcap \{\mathbf{P}^{\text{CL}}f(\mathcal{J}_i) \mid i \in I\} \subseteq \underline{Pf}(\mathcal{J});$$

applying c to both sides gives the desired

$$c\left(\bigcap \{\mathbf{P}^{\text{CL}}f(\mathcal{J}_i) \mid i \in I\}\right) \subseteq c(\underline{Pf}(\mathcal{J})) = c(\underline{Pf}(c(\mathcal{J}))) = c(\underline{Pf}(\mathcal{J})) = \mathbf{P}^{\text{CL}}f(\mathcal{J}).$$

The result for **AL** follows, since k establishes a natural isomorphism between $\mathbf{P}^{\mathbf{AL}}$ and the restriction of $\mathbf{P}^{\mathbf{CL}}$ to **AL**. ■

THEOREM 11. $\mathbf{P}^{\mathbf{CL}}$ preserves surjectivity of morphisms and projective limits (in the category **CL**). The same holds true for $\mathbf{P}^{\mathbf{AL}}$ in the category **AL**.

Armed with this theorem, one can apply to $\mathbf{P}^{\mathbf{CL}}$ (or $\mathbf{P}^{\mathbf{AL}}$) the theory of fixed-point constructions for functors as it is developed in Chapter IV of Gierz *et al.* (1980) and solve domain equations involving $\mathbf{P}^{\mathbf{CL}}$ in the category **CL** (or domain equations involving $\mathbf{P}^{\mathbf{AL}}$ in the category **AL**). We refer the reader to Gierz *et al.* (1980) for details (see esp. Scholium 4.9 in Chapter IV).

The rest of this section is devoted to the proof of Theorem 11. It will be based on the fact that $\mathbf{P}^{\mathbf{CL}}$ preserves adjunctions which is of great importance by itself. The reader should consult Gierz *et al.* (1980, 0, 3 and IV, 1) for the fundamental properties of adjunctions between posets. Here we merely summarize the few results that will be needed.

Let D and E be posets; a pair (f, g) of functions $f: D \rightarrow E$ and $g: E \rightarrow D$ is an *adjunction* between D and E if both f and g are monotone and $f(x) \geq y$ holds if and only if $x \geq g(y)$. f is the *upper adjoint* and g is the *lower adjoint*. One always has $g \circ f \leq 1_D$ and $f \circ g \geq 1_E$; conversely, the validity of these two inequalities for monotone f, g implies that (f, g) is an adjunction. Moreover, f is surjective $\Leftrightarrow g$ is injective $\Leftrightarrow f \circ g = 1_E$; similarly f is injective $\Leftrightarrow g$ is surjective $\Leftrightarrow g \circ f = 1_D$. One mapping in an adjunction pair uniquely determines the other; we denote the lower adjoint of f by \hat{f} . Any upper adjoint preserves infs and any lower adjoint preserves sups; the converse is true if D and E are complete lattices (i.e., any $f: D \rightarrow E$ preserving infs is an upper adjoint and any $g: E \rightarrow D$ preserving sups is a lower adjoint). Finally, if (f, g) is an adjunction between continuous lattices D and E then the following statements are equivalent:

- (1) f is continuous and preserves infs.
- (2) g (is continuous and) preserves sups and the relation \ll .

(Gierz *et al.* 1980, IV, Theorem 1.4.)

From here it follows that the category **CL** is dual to the category \mathbf{CL}^{op} of all continuous lattices and all continuous maps preserving sups and \ll (Gierz *et al.*, 1980, IV, Theorem 1.10).

From now until the end of this section, D and E will be continuous lattices and (f, g) an adjunction between D and E with f (and, of course, also g) continuous.

LEMMA 12. (f, g) is an adjunction between $P(D)$ and $P(E)$.

Proof. It is enough to show that $Y \leq_{\text{EM}} f(X) \Leftrightarrow g(Y) \leq_{\text{EM}} X$.

(1) Let $Y \leq_{\text{EM}} f(X)$; then for every $x \in X$ there is a $y \in Y$ (and also for every $y \in Y$ there is an $x \in X$) such that $y \leq f(x)$, i.e., $g(y) \leq x$. This shows $g(Y) \leq_{\text{EM}} X$.

(2) The proof of the converse implication is analogous. ■

LEMMA 13. $(P^{\text{CL}}f, P^{\text{CL}}g)$ is an adjunction between $\|P(D)\|_F$ and $\|P(E)\|_F$.

Proof. First note that f preserves infs and g preserves sups, so $P^{\text{CL}}f$ and $P^{\text{CL}}g$ are defined, by Lemma 9. Now

$$P^{\text{CL}}g \circ P^{\text{CL}}f \leq \bar{P}^{\text{CL}}(g \circ f) \leq \bar{P}^{\text{CL}}(1_D) = 1_{P^{\text{CL}}(D)},$$

and

$$P^{\text{CL}}f \circ P^{\text{CL}}g \geq \underline{P}^{\text{CL}}(f \circ g) \geq \underline{P}^{\text{CL}}(1_E) = 1_{P^{\text{CL}}(E)},$$

showing $(P^{\text{CL}}f, P^{\text{CL}}g)$ to be an adjunction. ■

It follows, in particular, that P^{CL} is a functor in the dual category \mathbf{CL}^{op} as well.

LEMMA 14. P^{CL} preserves surjectivity of morphisms in \mathbf{CL} .

Proof. If f is surjective then $f \circ \hat{f} = 1_E$ and hence $P^{\text{CL}}f \circ \widehat{P^{\text{CL}}f} = P^{\text{CL}}f \circ P^{\text{CL}}\hat{f} = 1_{P^{\text{CL}}(E)}$ by Theorem 8(iv), showing $P^{\text{CL}}f$ is surjective. ■

An analogous argument shows that P^{CL} preserves injectivity of morphisms in the dual category \mathbf{CL}^{op} . Before completing the proof of Theorem 11, we still need two technical lemmas.

LEMMA 15. $\underline{P}^{\text{CL}}$ and \bar{P}^{CL} are continuous on $[D \rightarrow E]$, for any continuous lattices D, E .

Proof. We give the proof for \bar{P}^{CL} ; $\underline{P}^{\text{CL}}$ is similar. Let $\{f_i \mid i \in I\}$ be a directed system where $f_i \in [D \rightarrow E]$ for all $i \in I$, and let $f = \sup\{f_i \mid i \in I\}$. We have to show that $\bar{P}^{\text{CL}}f(\mathcal{J}) = \sup\{\bar{P}^{\text{CL}}f_i(\mathcal{J}) \mid i \in I\}$ holds for all $\mathcal{J} \in \|P(D)\|_F$. The \geq -inequality is obvious since \bar{P}^{CL} is monotone (Theorem 8(v)). So let $Y \in \bar{P}^{\text{CL}}f(\mathcal{J})$; there exist Y_0, \dots, Y_m and $X_0, \dots, X_n \in \mathcal{J}$ such that $Y <_{\text{EM}} \{Y_0, \dots, Y_m\}^\uparrow$ and $Y_e <_{\text{EM}} f[\{X_0, \dots, X_n\}^\uparrow]$ for all $e \leq m$. Using the fact from the proof of Lemma 6, we have

$$(i) \quad (\forall j \in \text{Seq}) \quad (\exists y \in Y_e) \quad y \ll f(x^j) \text{ and}$$

$$(ii) \quad (\forall y \in Y_e) \quad (\exists i \leq n) \quad (\exists j \leq k_i) \quad y \ll f(x^j) \text{ whenever } j(i) = j$$

holding for all $e \leq m$ simultaneously.

Now $f(x^j) = \sup\{f_i(x^j) \mid i \in I\}$, for any $j \in \text{Seq}$. Since Seq and Y_e are finite and I is directed, there is some $i \in I$ such that (i) and (ii) hold with f replaced by f_i , for all $e \leq m$ simultaneously. But then $Y_e <_{\text{EM}} f_i[\{X_0, \dots, X_n\}^\uparrow]$, i.e., $Y_e \in \bar{P}f_i(\mathcal{J})$, for all $e \leq m$, and $Y \in \bar{P}^{\text{CL}}f_i(\mathcal{J})$ as desired. ■

LEMMA 16. *If f and g are continuous functions such that f preserves finite sups and g preserves finite infs, then*

$$\bar{P}^{\text{CL}}(g \circ f) = P^{\text{CL}}g \circ P^{\text{CL}}f$$

and

$$\underline{P}^{\text{CL}}(f \circ g) = P^{\text{CL}}f \circ P^{\text{CL}}g$$

(assuming the compositions are defined).

Proof. (1) Let $h = g \circ f$; $P^{\text{CL}}g \circ P^{\text{CL}}f \leq \bar{P}^{\text{CL}}h$ by Theorem 8(iv). Conversely, assume $Z \in \bar{P}^{\text{CL}}h(\mathcal{J})$; then $Z <_{\text{EM}} h[\{X_0, \dots, X_n\}^\uparrow]$ for some $X_0, \dots, X_n \in \mathcal{J}$, so

(i) $(\forall j \in \text{Seq}) (\exists z \in Z) z \ll h(x^j)$ and

(ii) $(\forall z \in Z) (\exists i \leq n) (\exists j \leq h_i) z \ll h(x^j)$ whenever $j(i) = j$. Let $Y_i = f(X_i)$ for $i \leq n$; since f preserves sups, we have $f(x^j) = f(x)^j = y^j$ and hence $h(x^j) = g(f(x^j)) = g(y^j)$. Replacing $h(x^j)$ by $g(y^j)$ in (i) and (ii) shows that $z <_{\text{EM}} g[\{Y_0, \dots, Y_n\}^\uparrow]$. But $Y_i = f(X_i) \in \bar{P}f(\mathcal{J})$ and so $Z \in \bar{P}g(\bar{P}f(\mathcal{J}))$. We have shown $\bar{P}^{\text{CL}}h(\mathcal{J}) \subseteq \bar{P}g(\bar{P}f(\mathcal{J}))$; applying c to both sides gives $\bar{P}^{\text{CL}}h(\mathcal{J}) \subseteq c(\bar{P}g(\bar{P}f(\mathcal{J}))) = c(\bar{P}g(c(\bar{P}f(\mathcal{J})))) = P^{\text{CL}}g(P^{\text{CL}}f(\mathcal{J}))$.

(2) We only need to show that $P^{\text{CL}}f \circ P^{\text{CL}}g \leq \underline{P}^{\text{CL}}(f \circ g)$. Let $Z \in P^{\text{CL}}f(P^{\text{CL}}g(\mathcal{J}))$; then $Z <_{\text{EM}} \{f(Y_0), \dots, f(Y_n)\}^\uparrow$, where $Y_0, \dots, Y_n \in P^{\text{CL}}g(\mathcal{J})$. Since g preserves finite infs, we have $Y_i \leq_{\text{EM}} g(X_i)$ for some $X_i \in \mathcal{J}$, $i \leq n$ (see the remark following the proof of Lemma 9); but then $Z <_{\text{EM}} \{f(g(X_0)), \dots, f(g(X_n))\}^\uparrow$, so $Z \in \underline{P}(f \circ g)(\mathcal{J})$. By applying c one gets $P^{\text{CL}}f(P^{\text{CL}}g(\mathcal{J})) \subseteq \underline{P}^{\text{CL}}(f \circ g)(\mathcal{J})$. ■

We conclude the proof of Theorem 11 with

LEMMA 17. P^{CL} preserves projective limits in the category \mathbf{CL} .

Proof. Let (J, \leq) be a directed poset, D_j , $j \in J$, continuous lattices, and g_{ij} , $i \leq j$, continuous inf-preserving maps from D_j into D_i such that $g_{ii} = 1_{D_i}$ and $g_{ij} \circ g_{jk} = g_{ik}$ holds whenever $i \leq j \leq k$. We denote the limit of this projective system in \mathbf{CL} by $\lim D$, and let $g_j: \lim D \rightarrow D_j$ be the accompanying limit maps; $g_{jk} \circ g_k = g_j$ whenever $j \leq k$. The following properties of limits are proved in Gierz *et al.* (1980, IV, Proposition 3.4):

$$(1) \quad g_j \circ \hat{g}_i = \sup\{g_{jk} \circ \hat{g}_{ik} \mid k \in J, i, j \leq k\} \text{ and}$$

(2) $\sup\{\hat{g}_j \circ g_j \mid j \in J\} = 1_{\lim D}$ (where the suprema are taken over directed sets of mappings).

Now $\mathbf{P}^{\text{CL}}D_j$, $j \in J$, and $\mathbf{P}^{\text{CL}}g_{ij}$, $i \leq j$, is also a projective system, with a limit $\lim \mathbf{P}^{\text{CL}}D$ and limit maps $h_j: \lim \mathbf{P}^{\text{CL}}D \rightarrow \mathbf{P}^{\text{CL}}D_j$. We have to prove that there is a unique isomorphism $f: \mathbf{P}^{\text{CL}}(\lim D) \rightarrow \lim \mathbf{P}^{\text{CL}}D$ such that $h_j \circ f = \mathbf{P}^{\text{CL}}g_j$ for all $j \in J$. By Theorem 3.11 in Gierz *et al.* (1980, IV), this is equivalent to showing

$$(i) \quad \mathbf{P}^{\text{CL}}g_j \circ \widehat{\mathbf{P}^{\text{CL}}g_j} = h_j \circ \hat{h}_j \text{ for all } j \in J \text{ and}$$

$$(ii) \quad \sup\{\widehat{\mathbf{P}^{\text{CL}}g_j} \circ \mathbf{P}^{\text{CL}}g_j \mid j \in J\} = 1_{\mathbf{P}^{\text{CL}}(\lim D)}.$$

(Theorem 3.11 in Gierz *et al.* (1980, IV) should have $1_{F(\lim D)}$ in place of $1_{\lim FD}$.)

$$\begin{aligned} (i) \quad & \mathbf{P}^{\text{CL}}g_j \circ \widehat{\mathbf{P}^{\text{CL}}g_j} \\ &= \mathbf{P}^{\text{CL}}g_j \circ \mathbf{P}^{\text{CL}}\hat{g}_j && \text{(by Lemma 13)} \\ &= \bar{\mathbf{P}}^{\text{CL}}(g_j \circ \hat{g}_j) && \text{(by Lemma 16)} \\ &= \bar{\mathbf{P}}^{\text{CL}}(\sup\{g_{jk} \circ \hat{g}_{jk} \mid k \in J, j \leq k\}) && \text{(by (1) above)} \\ &= \sup\{\bar{\mathbf{P}}^{\text{CL}}(g_{jk} \circ \hat{g}_{jk}) \mid k \in J, j \leq k\} && \text{(by Lemma 15)} \\ &= \sup\{\mathbf{P}^{\text{CL}}g_{jk} \circ \mathbf{P}^{\text{CL}}\hat{g}_{jk} \mid k \in J, j \leq k\} && \text{(by Lemma 16)} \\ &= \sup\{\mathbf{P}^{\text{CL}}g_{jk} \circ \widehat{\mathbf{P}^{\text{CL}}g_{jk}} \mid k \in J, j \leq k\} && \text{(by Lemma 13)} \\ &= h_j \circ \hat{h}_j && \text{(by (1) above applied} \\ & && \text{to the projective} \\ & && \text{system } \mathbf{P}^{\text{CL}}D_j; \mathbf{P}^{\text{CL}}g_{ij}). \end{aligned}$$

$$\begin{aligned} (ii) \quad & \sup\{\widehat{\mathbf{P}^{\text{CL}}g_j} \circ \mathbf{P}^{\text{CL}}g_j \mid j \in J\} \\ &= \sup\{\mathbf{P}^{\text{CL}}\hat{g}_j \circ \mathbf{P}^{\text{CL}}g_j \mid j \in J\} && \text{(by Lemma 13)} \\ &= \sup\{\underline{\mathbf{P}}^{\text{CL}}(\hat{g}_j \circ g_j) \mid j \in J\} && \text{(by Lemma 16)} \\ &= \underline{\mathbf{P}}^{\text{CL}}(\sup\{\hat{g}_j \circ g_j \mid j \in J\}) && \text{(by Lemma 15)} \\ &= \underline{\mathbf{P}}^{\text{CL}}(1_{\lim D}) && \text{(by (2) above)} \\ &= 1_{\mathbf{P}^{\text{CL}}(\lim D)}. \quad \blacksquare \end{aligned}$$

REFERENCES

- FRINK, O. (1954), Ideals in partially ordered sets, *Amer. Math. Monthly* **61**, 223–234.
- GIERZ, G., HOFMANN, K. H., KIEMEL, K., LAWSON, J. D., MISLOVE, M., AND SCOTT, D. S. (1980), “A Compendium of Continuous Lattices,” Springer-Verlag, New York/Berlin.
- GIERZ, G., AND KEIMEL, K. (1981), Continuous ideal completions and compactifications, in “Continuous Lattices,” Lecture Notes in Math. Vol. 871, pp. 97–124, Springer-Verlag, New York/Berlin.
- GRÄTZER, G. (1979), “Universal Algebra,” 2nd ed., Springer-Verlag, Berlin/New York.
- HRBACEK, K. (1985), Powerdomains as algebraic lattices, in “ICALP 85, Nafplion, Greece, July 1985” (W. Brauer, Ed.), Lecture Notes in Comput. Sci. Vol. 194 pp. 281–289, Springer-Verlag, New York/Berlin.
- HRBACEK, K. Convex powerdomains II, submitted.
- KAMIMURA, T., AND TANG, A. (1984), “Finitely Continuous Posets,” Technical Report TR-84-1, Dept. of Computer Science, University of Kansas, Lawrence, Kansas.
- MARKOWSKY, G., AND ROSEN, B. K. (1976), Bases for chain-complete posets, *IBM J. Res. Dev.* **20**, 138–147.
- PLOTKIN, G. (1976), A powerdomain construction, *SIAM J. Comput.* **5**, 452–486.
- PLOTKIN, G. (1981), unpublished 1981/82 lecture notes.
- SCOTT, D. S. (1970), Outline of a mathematical theory of computation, in “4th Annual Princeton Conference on Information Science and Systems,” pp. 169–176.
- SCOTT, D. S. (1972), Continuous lattices, in “Toposes, Algebraic Geometry and Logic” (F. W. Lawvere, Ed.), Lecture Notes in Math. Vol. 274, pp. 97–136, Springer-Verlag, New York/Berlin.
- SCOTT, D. S. (1976), Data types as lattices, *SIAM J. Comput.* **5**, 522–587.
- SCOTT, D. S. (1981), Lectures on a Mathematical Theory of Computation, Oxford Univ. Computing Laboratory Prog. Research Group-19, 148 pp.
- SCOTT, D. S. (1982), Domains for denotational semantics, in “ICALP 82, Aarhus, Denmark, July 1982” (M. Nielsen and E. M. Schmidt, Eds.), Lecture Notes in Comput. Sci. Vol. 140, pp. 577–613, Springer-Verlag, New York/Berlin.
- SMYTH, M. B. (1978), Powerdomains, *J. Comput. System Sci.* **16**, 23–36.
- SMYTH, M. B. (1983), The largest cartesian closed category of domains, *Theor. Comput. Sci.* **27**, 109–119.
- ERNÉ, M. (1981), A completion-invariant extension of the concept of continuous lattices, in “Continuous Lattices” (B. Banaschewski and R.-E. Hoffman, Eds.), Lecture Notes in Math. Vol. 871, pp. 45–60, Springer-Verlag, Berlin/New York.